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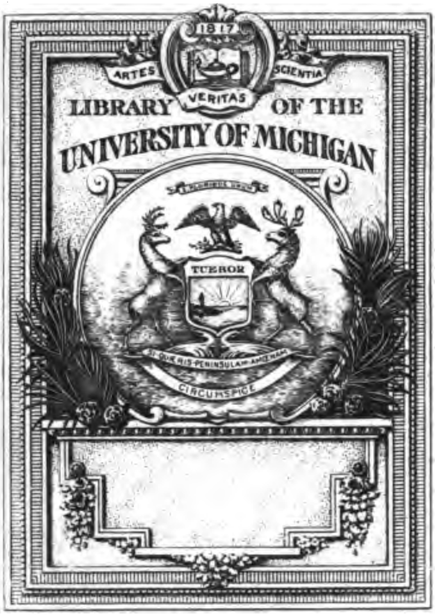

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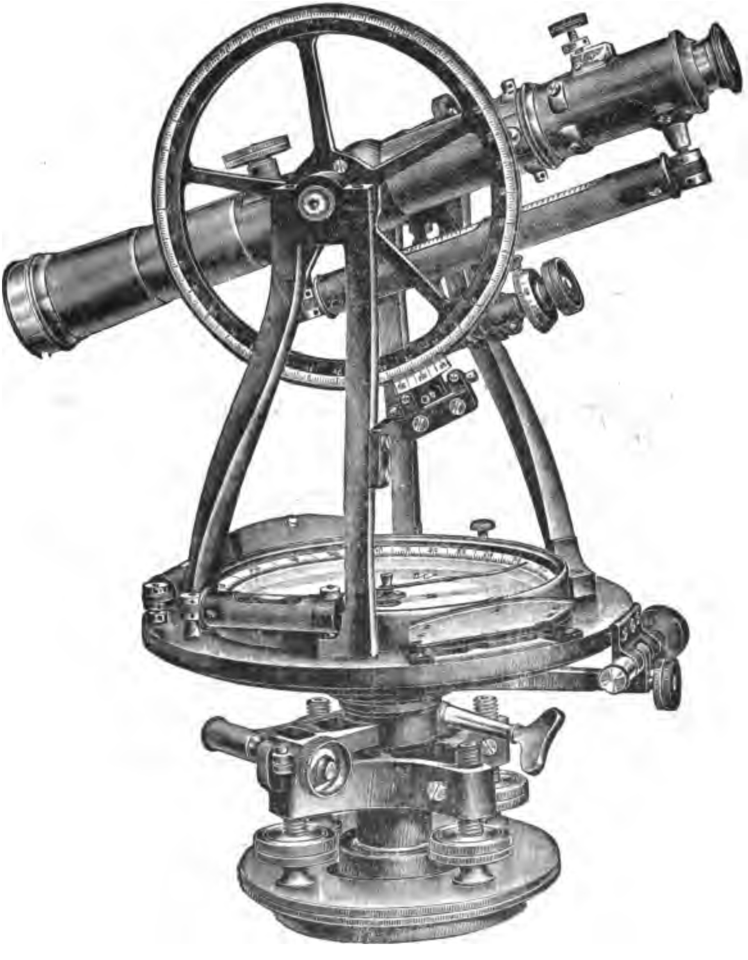

SURVEYOR'S TRANSIT

(See description, page 36)

UNIV. OF MICH.

PLANE TRIGONOMETRY

BY

EDWARD R. utledge ROBBINS

FORMERLY OF LAWRENCEVILLE SCHOOL

H L Selmeier
Eng 8
Trig 1

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PREFACE

Much that is superfluous to a complete scientific understanding of Trigonometry has crept into the classroom and the text book in recent years, and the time allotted to the study has been insufficient for the class to master it all. In the attempt there has too often resulted a superficial acquaintance with a mass of principles, but not a thorough familiarity with the essential truths. Nor has the usual course always produced a satisfactory skill in operating with these new processes and with the instruments of calculation that Trigonometry employs.

This book is the product of an experience of fifteen years in teaching the subject, and presents the essential elements of Trigonometry in a manner that is believed to be both sound and teachable. The order of development of the science is substantially the same as that of our best texts, but only such material as is necessary to a complete understanding and knowledge of the subject has been incorporated in its treatment.

Principles are developed and their application to problems is made immediately; formulas are derived and they are made of real value to the pupil by instant use; difficulties are foreseen and the mind of the learner is made the more ready to overcome them; correlated formulas are grouped and are invariably succeeded by exercises employing them in their solution. This method of treatment increases the power of the pupil when he attacks the greatest of trigonometric difficulties, the identity, because he has acquired the necessary skill, little by little, in applying the formulas that he has learned, a few at a time.

The ultimate purpose has been to present sufficient drill work of such character as to meet the diverse college and university requirements. Angles are uniformly expressed in the usual units. But if either the teacher or the university requires that they be expressed in degrees and the decimal part of a degree, the reduction is so simple, and the additional labor so trifling, that no inconvenience is likely to result.

The order of the four cases of oblique triangles that has been followed differs from the usual arrangement in that the ambiguous case is placed last. Emphasis is given to the "scheme" of all logarithmic work, and the learner is urged to prepare this from beginning to end, in every example, before commencing the actual logarithmic computation.

Attention is particularly invited to the arrangement, location, number, and variety of the examples, and to the excellence of the geometrical diagrams.

EDWARD R. ROBBINS.

PHILADELPHIA, PENNSYLVANIA.

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REFERENCES TO THE PLANE GEOMETRY

The presentation and the study of Plane Trigonometry assume a knowledge of Plane Geometry. Consciously or unconsciously the student is continually using truths acquired in Geometry. Definite citation for every such occurrence, of the page or article in the Geometry, of the truth employed in the Trigonometry, would become both laborious and unnecessary. But in consistent, logical reasoning the citation is imperative, and the author cites in this book theorems, etc., from his Plane Geometry by the number of the paragraph. Those thus specifically cited are given here for the convenience of the learner.

Ax. 1. Magnitudes that are equal to the same thing, or to equals, are equal to each other.

27. Homologous parts of equal figures are equal.

53. Two right triangles are equal if the two legs of one are equal respectively to the two legs of the other.

55. The angles opposite the equal sides of an isosceles triangle are equal.

72. Two right triangles are equal if the hypotenuse and an adjoining angle of one are equal respectively to the hypotenuse and an adjoining angle of the other.

73. Two right triangles are equal if the hypotenuse and a leg of one are equal respectively to the hypotenuse and a leg of the other.

74. Two right triangles are equal if a leg and the adjoining acute angle of one are equal respectively to a leg and the adjoining acute angle of the other.

97. If a transversal intersects two parallels, the alternate interior angles are equal.

110. The sum of the angles of any triangle is two right angles; that is, 180°.

114. The acute angles of a right triangle are complementary.

115. Each angle of an equiangular triangle is 60°.

122. If two sides of a triangle are unequal, the angle opposite the greater side is greater than the angle opposite the less side.

123. If two angles of a triangle are unequal, the side opposite the greater angle is greater than the side opposite the less angle.

124. The hypotenuse is the longest side of a right triangle.

215. The line perpendicular to a radius at its extremity is tangent to the circle.

237. To **measure a quantity** is to find the number of times it contains another quantity of the same kind, called the **unit.** This *number* is the ratio of the quantity to the unit.

242. If two variables are always equal and each approaches a limit, their limits are equal

244. In the same circle (or in equal circles) **the ratio of two central angles is equal to the ratio of their intercepted arcs.**

245. A central angle is measured by its intercepted arc.

292. In any proportion the terms are also in proportion by alternation (that is, the first term is to the third as the second is to the fourth).

296. In any proportion the terms are also in proportion by composition and division (that is, the sum of the first two terms is to their difference as the sum of the last two terms is to their difference).

312. Triangles are **similar** if they are mutually equiangular and their homologous sides are proportional.

315. Two right triangles are similar if an acute angle of one is equal to an acute angle of the other.

321. If two triangles have their homologous sides perpendicular, they are similar.

323, 3. **In similar figures homologous sides are proportional.**

343. The sum of the squares of the legs of a right triangle is equal to the square of the hypotenuse.

344. The square of either leg of a right triangle is equal to the square of the hypotenuse minus the square of the other leg.

345. In an obtuse triangle the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides plus twice the product of one of these two sides and the projection of the other side upon that one.

346. In any triangle the square of the side opposite an acute angle is equal to the sum of the squares of the other two sides minus twice the product of one of these two sides and the projection of the other side upon that one.

378. The area of a triangle is equal to half the product of its base by its altitude.

444. Let $C =$ circumference and $R =$ radius. Then, $\boldsymbol{C = 2\pi R}$.

PLANE TRIGONOMETRY

CHAPTER I

FUNCTIONS OF AN ACUTE ANGLE

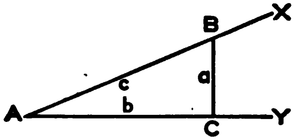
1. Function. If the value of one quantity depends upon the value of another, the first quantity is called a **function** of the second.

ILLUSTRATIONS. The time required by a man to walk a given distance depends upon his velocity. Thus the time is a function of the velocity. The circumference of a circle depends upon the length of its radius. Thus the circumference is a function of the radius. Likewise, the length of the arc intercepted by a central angle in a given circle is a function of the central angle.

2. Six ratios concerning an acute angle. If from any point in one side of an acute angle A, a perpendicular is drawn to the other side, in the right triangle formed:

$$\textbf{The Sine} \quad \text{of } \angle A = \frac{\text{the opposite leg}}{\text{the hypotenuse}}.$$

$$\textbf{The Cosine} \text{ of } \angle A = \frac{\text{the adjacent leg}}{\text{the hypotenuse}}.$$

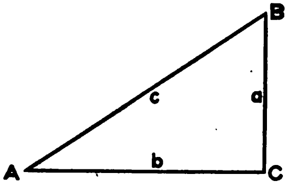


The **Tangent** of $\angle A = \dfrac{\text{the opposite leg}}{\text{the adjacent leg}}$.

The **Cotangent** of $\angle A = \dfrac{\text{the adjacent leg}}{\text{the opposite leg}}$.

The **Secant** of $\angle A = \dfrac{\text{the hypotenuse}}{\text{the adjacent leg}}$.

The **Cosecant** of $\angle A = \dfrac{\text{the hypotenuse}}{\text{the opposite leg}}$.



It is customary to denote the side opposite the $\angle A$ by "a," the side opposite the $\angle B$, or the adjacent leg, by "b," and the hypotenuse by "c." These definitions are thereby abbreviated thus:

$$\sin A = \frac{a}{c};\quad \tan A = \frac{a}{b};\quad \sec A = \frac{c}{b};$$

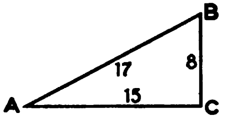
$$\cos A = \frac{b}{c};\quad \cot A = \frac{b}{a};\quad \csc A = \frac{c}{a}.$$

Perfect familiarity with these fundamental definitions is essential to intelligent progress in this science.

EXAMPLE. If in a right triangle $a = 8$, $b = 15$, $c = 17$, write the ratios of $\angle A$ and of $\angle B$.

From the definitions of the six ratios

$\sin A = \frac{8}{17}$; $\tan A = \frac{8}{15}$; $\sec A = \frac{17}{15}$;
$\cos A = \frac{15}{17}$; $\cot A = \frac{15}{8}$; $\csc A = \frac{17}{8}$.
Also, $\sin B = \frac{15}{17}$; $\tan B = \frac{15}{8}$; $\sec B = \frac{17}{8}$;
$\cos B = \frac{8}{17}$; $\cot B = \frac{8}{15}$; $\csc B = \frac{17}{15}$.

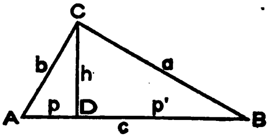


EXERCISE 1

Draw a right triangle *ABC* for each of the following six examples, assigning to each side its given value, and then write the values of the six ratios of each acute angle:

1. $a=3$; $b=4$; $c=5$.
3. $a=12$; $b=5$; $c=13$.
2. $a=8$; $b=6$; $c=10$.
4. $a=2$; $b=6$; $c=2\sqrt{10}$.
5. $a=1$; $b=2\sqrt{6}$; $c=5$
6. $a=2mn$; $b=m^2-n^2$; $c=m^2+n^2$.

In the adjoining figure *ABC*, *ADC*, *BCD*, are right triangles.



7. In $\triangle ABC$ write the values of the six ratios of $\angle A$; of $\angle B$.

8. In $\triangle ADC$ write the values of the six ratios of $\angle A$; of $\angle ACD$.

9 In $\triangle BCD$ write the values of the six ratios of $\angle B$; of $\angle BCD$.

10. Show that in a right triangle *ABC*

$a = c \cdot \sin A$; $a = b \cdot \tan A$; $c = b \cdot \sec A$;

$b = c \cdot \cos A$; $b = a \cdot \cot A$; $c = a \cdot \csc A$

11. Show that in a right triangle *ABC*

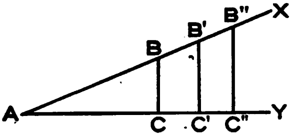
$b = c \cdot \sin B$; $b = a \cdot \tan B$; $c = a \cdot \sec B$;

$a = c \cdot \cos B$; $a = b \cdot \cot B$; $c = b \cdot \csc B$.

3. Trigonometry is that science which treats of the relations of the six ratios, sine, cosine, tangent, cotangent, secant, and cosecant of angles, and also treats of the measurement of the magnitudes concerned in any triangle.

These ratios are called **trigonometric ratios.**

4. Theorem. Each trigonometric ratio of every acute angle has one value and only one.



Given: The angle A.

To Prove: $\sin A =$ constant; $\cos A =$ constant; $\tan A =$ constant; etc.

Proof: Take several points B, B', B'' on AX, and draw $\perp$s BC, $B'C'$, $B''C''$ to AY. Now $\triangle$s ABC, $AB'C'$, $AB''C''$ are similar triangles (Geometry, 315).

$$\therefore \frac{BC}{B'C'} = \frac{AB}{AB'} \text{ and } \frac{BC}{B''C''} = \frac{AB}{AB''} \quad \text{(Geom. 312).}$$

$$\therefore \frac{BC}{AB} = \frac{B'C'}{AB'} \text{ and } \frac{BC}{AB} = \frac{B''C''}{AB''} \quad \text{(Alternation, Geom. 292).}$$

$$\therefore \frac{BC}{AB} = \frac{B'C'}{AB'} = \frac{B''C''}{AB''} \quad \text{(Ax. 1).}$$

But $\sin A = \dfrac{BC}{AB}$, or $\sin A = \dfrac{B'C'}{AB'}$, or $\sin A = \dfrac{B''C''}{AB''}$ (Def., § 2).

But these ratios are all equal.

$\therefore$ $\sin A =$ constant wherever the $\perp$ is drawn.

Similarly, the other functions of $\angle A$ are proved constant.

Note. If one of the $\perp$s is drawn from a point in AY $\perp$ to AX, the triangle formed is still similar to the others, and the proof is unchanged.

5. Theorem. For any possible value that a trigonometric ratio may have there is only one acute angle.

Proof: Let the given value be reduced to its simplest form. The value of any trigonometric ratio, reduced to its simplest form, gives the lengths of two sides of a certain right triangle. But if two sides of a right triangle are known, the triangle is determined (Geom. 53, 73). Hence there can be only one acute angle corresponding to each value of any trigonometric ratio.

6. Trigonometric ratios are **functions** of angles. This is true because the value of each ratio depends upon the value of the angle.

7. Limitations to the values of the trigonometric functions of an acute angle. The hypotenuse is the longest side of a right triangle (Geom. 124). Hence the value of sin A, which is $\frac{a}{c}$, cannot be greater than unity.

The value of cos A, which is $\frac{b}{c}$, cannot be greater than unity.

The value of tan A, which is $\frac{a}{b}$, has no limitation.

The value of cot A, which is $\frac{b}{a}$, has no limitation.

The value of sec A, which is $\frac{c}{b}$, cannot be less than unity.

The value of csc A, which is $\frac{c}{a}$, cannot be less than unity.

Thus sin A could not equal 2, or $\frac{3}{2}$, or $\sqrt{2}$, or $\sqrt{5}$.

And sec A could not equal $\frac{1}{2}$, or $\frac{2}{3}$, or .758, or $\sqrt{.3}$.

But tan A and cot A could equal any of these values.

NOTE. The truths of this paragraph justify the use of the word "possible" in the theorem of § 5.

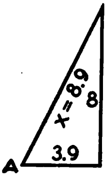
8. If any two sides of a right triangle are known, it is possible to find the remaining side (Geom. 343, 344).

EXAMPLE. If the legs of a right triangle are 8 and 3.9, find the hypotenuse and write the six functions of one of the acute angles of the triangle.

Draw a right △ and denote its hypotenuse by x; the legs are 8 and 3.9.

$$\therefore x = \sqrt{8^2 + 3.9^2} = \sqrt{79.21} = 8.9.$$

$$\therefore \sin A = \frac{8}{8.9} = \frac{80}{89}; \cos A = \frac{3.9}{8.9} = \frac{39}{89}; \text{ etc.}$$



EXERCISE 2

Find the remaining side of each of the following right triangles, and write the functions of $\angle A$.

1. $a=9$; $b=12$.
2. $c=13$; $a=5$.
3. $b=5.5$; $c=7.3$.
4. $c=29$; $a=21$.
5. $a=1$; $b=3$.
6. $c=10$; $a=5$.
7. $b=2\sqrt{14}$; $c=8$.
8. $a=\sqrt{p^2+q^2}$; $b=\sqrt{2pq}$.
9. $a=m$; $b=n$.
10. $a=x-1$; $c=x+1$.

9. Designation of angles. Any letter may be used to represent the value of an angle. Certain Greek letters are frequently employed in this way. Those usually occurring are the following:

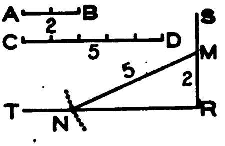
α (Alpha)
β (Beta)
γ (Gamma)
δ (Delta)
θ (Theta)
κ (Kappa)
λ (Lambda)
μ (Mu)
ν (Nu)
π (Pi)
ϕ (Phi)
ψ (Psi)

10. Problem. To construct the acute angle if the value of any of its functions is known.

Given: $\sin x = \frac{2}{5}$.

Required: To construct $\angle x$.

Construction: Take two lines, $AB=2$ units and $CD=5$ units. Construct a right $\angle SRT$. On RS take $RM=AB$, and using M as a center and CD as a radius, describe an arc cutting RT at N. Draw MN.



Statement: $\angle MNR = \angle x$.

Proof: $\sin x = \frac{2}{5}$ (Given); $\sin MNR = \frac{2}{5}$ (Const. and Def.).

$\therefore \angle MNR = \angle x$.

Likewise, the construction is accomplished if the value of any other function is given.

EXERCISE 3

Construct the angle in each of the following equations:

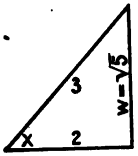
1. $\cos x = \frac{3}{4}$.	4. $\cot A = \frac{7}{8}$.	7. $\cos y = \frac{1}{8}$.	10. $\tan \theta = .3$.
2. $\tan y = \frac{1}{5}$.	5. $\sin \beta = \frac{3}{8}$.	8. $\sec M = 2.6$.	11. $\cot \gamma = 1.5$.
3. $\sec \theta = \frac{5}{4}$.	6. $\csc \phi = 4$.	9. $\sin 30° = \frac{1}{2}$.	12. $\csc x = 1\frac{1}{8}$.

11. Problem. To find the remaining functions of an angle if one function is given.

Ex. 1. **Given :** $\sec x = 1.5 \left(= \frac{3}{2}\right)$.

Required : The values of the remaining functions of $\angle x$.

Solution: Construct a right triangle whose hypotenuse = 3 units, and one of whose legs = 2 units. Denote the included angle by x, and the opposite leg by w.



Then $w = \sqrt{3^2 - 2^2} = \sqrt{5}$.

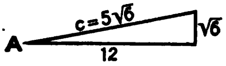
$$\therefore \sin x = \frac{\sqrt{5}}{3};\ \cos x = \frac{2}{3};\ \tan x = \frac{\sqrt{5}}{2};$$

$$\cot x = \frac{2}{\sqrt{5}} = \frac{2}{5}\sqrt{5};\ \csc x = \frac{3}{\sqrt{5}} = \frac{3}{5}\sqrt{5}.$$

Ex. 2. **Given :** $\tan A = \frac{1}{12}\sqrt{6}\left(= \frac{\sqrt{6}}{12}\right)$.

Required : The values of the remaining functions of $\angle A$.

Solution: Denote an acute $\angle$ of a right Δ by A, the opposite leg by $\sqrt{6}$, the adjacent leg by 12, and the hypotenuse by c.



Then $c = \sqrt{144 + 6} = \sqrt{150} = 5\sqrt{6}$.

$$\therefore \sin A = \frac{\sqrt{6}}{5\sqrt{6}} = \frac{1}{5};\ \cos A = \frac{12}{5\sqrt{6}} = \frac{2\sqrt{6}}{5};$$

$$\cot A = \frac{12}{\sqrt{6}} = 2\sqrt{6};\ \text{etc.}$$

Note. It is obvious that in § 10 only an angle is required, while in § 11 one of an indefinite number of similar triangles is employed in obtaining other ratios.

EXERCISE 4

Find the remaining functions of the given angle, if

1. $\sin A = \frac{4}{5}$.
2. $\cos x = \frac{2}{3}$.
3. $\tan \theta = 3$.
4. $\sec y = 5$.
5. $\cot M = 2$.
6. $\csc \beta = \sqrt{2}$.
7. $\cos \delta = \frac{1}{2}\sqrt{3}$.
8. $\sec a = 4$.
9. $\csc \phi = 2$.
10. $\tan A = 1\frac{1}{3}$.
11. $\sin \theta = \frac{1}{x}$.
12. $\cot \gamma = m$.
13. $\cos A = \frac{r}{s}$.
14. $\sin x = \frac{2}{m}$.
15. $\sec B = \frac{a+1}{a-1}$.
16. $\tan 60° = \sqrt{3}$.

ELEMENTARY RELATIONS OF THE FUNCTIONS

12. Reciprocal functions. From the definitions of § 2 it is apparent that:

The sine and cosecant of an angle are reciprocals.

The cosine and secant of an angle are reciprocals.

The tangent and cotangent of an angle are reciprocals.

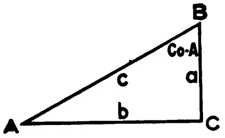
$$\therefore \sin A = \frac{1}{\csc A}, \text{ or } \csc A = \frac{1}{\sin A}, \text{ or } \sin A \cdot \csc A = 1;$$

$$\cos A = \frac{1}{\sec A}, \text{ or } \sec A = \frac{1}{\cos A}, \text{ or } \cos A \cdot \sec A = 1;$$

$$\tan A = \frac{1}{\cot A}, \text{ or } \cot A = \frac{1}{\tan A}, \text{ or } \tan A \cdot \cot A = 1.$$

EXERCISE 5

1. If $\sin A = \frac{2}{7}$, $\csc A = ?$
2. If $\cos x = \frac{1}{2}$, $\sec x = ?$
3. If $\cot \theta = 5$, $\tan \theta = ?$
4. If $\sec a = \sqrt{5}$, $\cos a = ?$
5. If $\tan M = x$, $\cot M = ?$
6. If $\sin 60° = \frac{1}{2}\sqrt{3}$, $\csc 60° = ?$
7. If $\cos 45° = \frac{1}{2}\sqrt{2}$, $\sec 45° = ?$
8. If $\csc 30° = 2$, $\sin 30° = ?$



13. Complementary functions. In the right $\triangle ABC$ the acute angles are complementary (Geom. 114). That is, $\angle B =$ the complement of $\angle A$. This is briefly expressed thus, $B = \text{co-}A$.

Now $\sin B = \sin (\text{co-}A) = \frac{b}{c}$. But $\cos A = \frac{b}{c}$.

$$\therefore \sin (\text{co-}A) = \cos A.$$

That is, **the *sine* of the complement of an angle = the *cosine* of the angle.**

Likewise, **the *tangent* of the complement of an angle = the *cotangent* of the angle,**

and **the *secant* of the complement of an angle = the *cosecant* of the angle.**

Conversely, $\cos B = \cos (\text{co-}A) = \frac{a}{c}$. But $\sin A = \frac{a}{c}$.

$$\therefore \cos (\text{co-}A) = \sin A.$$

That is, **the *cos* of the comp. of an angle = the *sine* of the angle.**

Likewise, **the *cot* of the comp. of an angle = the *tan* of the angle,**

and **the *csc* of the comp. of an angle = the *sec* of the angle.**

Thus one understands that the names co-sine, co-tangent, co-secant are abbreviations of the fuller form: sine of the complement, tangent of the complement, secant of the complement.

EXERCISE 6

Express each of the following as a function of the complementary angle:

1. $\sin 43°$.
2. $\cos 75°$.
3. $\sec 13°$.
4. $\tan 35°$.
5. $\cot 26°$.
6. $\csc 80°$.
7. $\sin 30°$.
8. $\cos 45°$.
9. $\sin (90° - A)$.
10. $\cot (90° - x)$.
11. $\cos \theta$.
12. $\tan N$.
13. $\sin 50° 30'$.
14. $\sec 68° 45'$.
15. $\cos 10° 10'$.

16. $\tan 72° 36'$. **19.** $\cos (90° - \theta)$. **22.** $\sin 1°$.

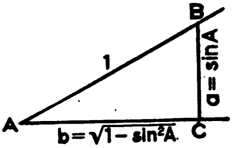
17. $\csc 42° 42'$. **20.** $\tan (90° - \beta)$. **23.** $\cot 60°$.

18. $\cot 82° 58'$. **21.** $\sec 89°$. **24.** $\cos (80° + x)$.

NOTE. The student should remember that "sin A," "cos A," etc., are mere symbols for a number, and we may operate with them precisely as we do with other numbers. Thus $(\sin A)(\sin A) = (\sin A)^2$ and means that this particular function is squared. Hence $(\sin A)^2$ is written $\sin^2 A$, the exponent belonging to and affecting the *function* and *not* the *angle*.

14. PROBLEM. To express all of the functions of an acute angle in terms of the sine of the angle.

SOLUTION: On one side of $\angle A$ take AB = a unit of length; draw $BC \perp$ to the other side of the $\angle$.



$\sin A = \frac{a}{1}$. $\therefore a = \sin A$;

$$b = \sqrt{1 - a^2} = \sqrt{1 - \sin^2 A}.$$

$$\therefore \cos A = \frac{\sqrt{1 - \sin^2 A}}{1} = \sqrt{1 - \sin^2 A}; \quad \tan A = \frac{\sin A}{\sqrt{1 - \sin^2 A}};$$

$$\cot A = \frac{\sqrt{1 - \sin^2 A}}{\sin A}; \quad \csc A = \frac{1}{\sin A}; \quad \sec A = \frac{1}{\sqrt{1 - \sin^2 A}}.$$

Thus all five of the functions of A are expressed in terms of $\sin A$.

EXERCISE 7

Express all of the remaining functions of angle A in terms

1. Of the cosine of A.
2. Of the tangent of A.
3. Of the cotangent of A.
4. Of the secant of A.
5. Of the cosecant of A.

NOTE. This paragraph shows that the six functions are not independent, and that each can be expressed in terms of any other.

15. PROBLEM. To derive the five formulas :

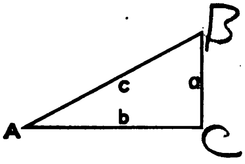
(a) $\sin^2 A + \cos^2 A = 1.$

(c) $\cot^2 A + 1 = \csc^2 A.$

(b) $\tan^2 A + 1 = \sec^2 A.$

(d) $\tan A = \dfrac{\sin A}{\cos A}.$

(e) $\cot A = \dfrac{\cos A}{\sin A}.$



Proofs:

(a) $\sin^2 A + \cos^2 A = \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2$

$= \frac{a^2}{c^2} + \frac{b^2}{c^2}$

$= \frac{a^2 + b^2}{c^2}$

$= \frac{c^2}{c^2}$ (Geom. 343)

$= 1.$

$\therefore \sin^2 A + \cos^2 A = 1.$

By transposition, etc., we obtain

$\sin^2 A = 1 - \cos^2 A.$

$\therefore \sin A = \sqrt{1 - \cos^2 A}.$

Also, $\cos^2 A = 1 - \sin^2 A.$

$\therefore \cos A = \sqrt{1 - \sin^2 A}.$

(b) $\tan^2 A + 1 = \left(\frac{a}{b}\right)^2 + 1$

$= \frac{a^2 + b^2}{b^2}$

$= \frac{c^2}{b^2}$ (Geom. 343)

$= \sec^2 A.$

$\therefore \tan^2 A + 1 = \sec^2 A.$

By transposition, etc.,

$\tan^2 A = \sec^2 A - 1;$

$\therefore \tan A = \sqrt{\sec^2 A - 1}.$

$\sec^2 A = \tan^2 A + 1;$

$\therefore \sec A = \sqrt{\tan^2 A + 1}.$

Also, $\sec^2 A - \tan^2 A = 1.$

(c) $\cot^2 A + 1 = \left(\frac{b}{a}\right)^2 + 1$

$= \frac{b^2 + a^2}{a^2}$

$= \frac{c^2}{a^2}$ (Geom. 343)

$= \csc^2 A.$

$\therefore \cot^2 A + 1 = \csc^2 A.$

By transposition, etc.,

$\cot^2 A = \csc^2 A - 1;$

$\therefore \cot A = \sqrt{\csc^2 A - 1}$

$\csc^2 A = \cot^2 A + 1;$

$\therefore \csc A = \sqrt{\cot^2 A + 1}.$

Also, $\csc^2 A - \cot^2 A = 1.$

(d) $\frac{\sin A}{\cos A} = \frac{\frac{a}{c}}{\frac{b}{c}}$

$= \frac{a}{b}$

$= \tan A.$

$\therefore \tan A = \frac{\sin A}{\cos A}.$

(e) $\frac{\cos A}{\sin A} = \frac{\frac{b}{c}}{\frac{a}{c}}$

$= \frac{b}{a}$

$= \cot A.$

$\therefore \cot A = \frac{\cos A}{\sin A}.$

16. Perhaps the most satisfactory method of establishing relations between the trigonometric functions is that employed in the preceding paragraph. This consists in reducing one member of the relation to be established, by known substitutions, until the other member is obtained. This method is not always most convenient nor most simple, but it possesses advantages over other methods, as a guide, and it is the safest for beginners. Briefly, consider one member of the equation as the example, or expression to be reduced, and the other member as the answer. All of the preceding formulas should be thoroughly memorized.

Ex. 1. **Prove:** $\sin A \cdot \sec A \cdot \cot A = 1.$

Proof: $\sin A \cdot \sec A \cdot \cot A = \sin A \cdot \frac{1}{\cos A} \cdot \frac{\cos A}{\sin A}$ (By simple substitutions.)

$= 1$, **result.** (By cancellation.) Q. E. D.

Ex. 2. **Prove:** $\sec x - \cos x = \sin x \tan x$.

Proof:

$$\begin{aligned}\sec x - \cos x &= \frac{1}{\cos x} - \cos x\\ &= \frac{1-\cos^2 x}{\cos x}\\ &= \frac{\sin^2 x}{\cos x}\\ &= \frac{\sin x}{1}\cdot\frac{\sin x}{\cos x}\\ &= \sin x\cdot\tan x, \text{ result.}\end{aligned}$$

Q. E. D.

Ex. 3. Prove: $\dfrac{\cos\theta}{1-\sin\theta} = \dfrac{1+\sin\theta}{\cos\theta}$.

Proof: Notice that the left denominator would be part of a formula if multiplied by $1 + \sin\theta$.

$$\begin{aligned}\frac{\cos\theta}{1-\sin\theta} &= \frac{\cos\theta}{1-\sin\theta}\cdot\frac{1+\sin\theta}{1+\sin\theta} = \frac{\cos\theta(1+\sin\theta)}{1-\sin^2\theta}\\ &= \frac{\cos\theta(1+\sin\theta)}{\cos^2\theta}\\ &= \frac{1+\sin\theta}{\cos\theta}, \text{ result.}\end{aligned}$$

Q. E. D.

EXERCISE 8

By means of formulas already established, prove:

1. $\sin A \sec A = \tan A$.
2. $\sin A \cot A = \cos A$.
3. $\cos A \csc A \tan A = 1$.
4. $\sec x - \tan x \sin x = \cos x$.
5. $\csc\theta - \cot\theta\cos\theta = \sin\theta$.
6. $\csc\alpha - \sin\alpha = \cos\alpha\cot\alpha$.
7. $\tan\beta + \cot\beta = \sec\beta\csc\beta$.
8. $\cos M\sqrt{\sec^2 M - 1} = \sin M$.
9. $\sec^2 y + \csc^2 y = \sec^2 y \csc^2 y$.
10. $(\sqrt{1-\sin A} + \sqrt{1+\sin A})^2 = 2(1+\cos A)$.
11. $\dfrac{1}{\sec x - \tan x} - \dfrac{1}{\sec x + \tan x} = 2\tan x$.

12. $\dfrac{\sin\theta}{1+\cos\theta}=\dfrac{1-\cos\theta}{\sin\theta}.$ 13. $\dfrac{\tan\lambda}{\sec\lambda-1}=\dfrac{\sec\lambda+1}{\tan\lambda}.$

14. $\dfrac{1+\sin x}{1-\sin x}-\dfrac{1-\sin x}{1+\sin x}=4\tan x\sec x.$

15. $\dfrac{1+\cos\phi}{1-\cos\phi}-\dfrac{1-\cos\phi}{1+\cos\phi}=4\cot\phi\csc\phi.$

attention

16. $\sqrt{\dfrac{1-\cos A}{1+\cos A}}=\dfrac{1-\cos A}{\sin A}.$ 17. $\sqrt{\dfrac{1-\cos A}{1+\cos A}}=\dfrac{\sin A}{1+\cos A}.$

18. $\sin^2 x-\cos^2 y=\sin^2 y-\cos^2 x.$

19. $\cos^2 x-\cos^2 y=\sin^2 y-\sin^2 x.$

20. $\sec^2 x+\tan^2 y=\tan^2 x+\sec^2 y.$

21. $\csc^2 A-\csc^2 B=\cot^2 A-\cot^2 B.$

22. $\sin^2 A\cos^2 B-\cos^2 A\sin^2 B=\sin^2 A-\sin^2 B=\cos^2 B-\cos^2 A$

23. $(\tan\theta+\cot\theta)\sin\theta\cos\theta=1.$

24. $\cos^2 A-\sin^2 A=2\cos^2 A-1=1-2\sin^2 A.$

25. $(\sin A+\cos A)^2+(\sin A-\cos A)^2=2.$

26. $\dfrac{\tan m+\tan n}{\cot m+\cot n}=\tan m\tan n.$

27. $\dfrac{\cot x+\tan y}{\tan x+\cot y}=\cot x\tan y.$

28. $\dfrac{1-\sin A}{1+\sin A}=(\sec A-\tan A)^2.$

29. $\dfrac{1+\cos x}{1-\cos x}=(\csc x+\cot x)^2.$

30. $(\sin x+\cos x)(\tan x+\cot x)=\sec x+\csc x.$

31. $\sin x(\sec x+\csc x)-\cos x(\sec x-\csc x)=\sec x\csc x.$

32. $\cot A-\sec A\csc A(1-2\sin^2 A)=\tan A.$

THE FUNCTIONS OF 45°, 30°, 60°, 0°, 90°

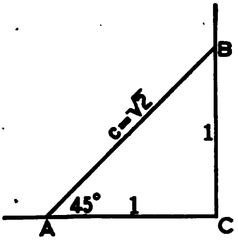
17. Problem. To find the numerical values of the functions of 45°.

Solution: Construct a right $\angle C$ and measure on its sides CA and CB, each a unit of length.

Draw AB and denote its length by c.

Now $c=\sqrt{1^2+1^2}=\sqrt{2}$ (Geom. 343).

Also $\angle A=45°$ (Geom. 55; 114).

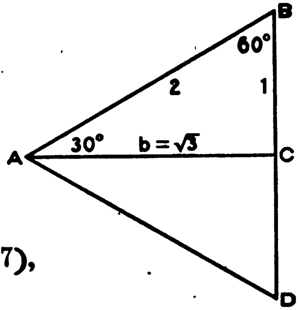


$\therefore\ \sin 45°=\dfrac{1}{\sqrt{2}}=\frac{1}{2}\sqrt{2}$; $\cot 45°=1$;

$\cos 45°=\dfrac{1}{\sqrt{2}}=\frac{1}{2}\sqrt{2}$; $\sec 45°=\sqrt{2}$;

$\tan 45°=1$; $\csc 45°=\sqrt{2}$.

18. Problem. To find the numerical values of the functions of 30° and of 60°.



Solution: Construct an equilateral $\triangle ABD$, each side being two units in length. From any vertex, A, draw $AC \perp$ to the opposite side forming the right $\triangle$s. These are equal (Geom. 73).

Also $\angle B=60°$ (Geom. 115).

$\therefore\ \angle BAC=\angle DAC=30°$ (Geom. 27),

and $BC=CD=1$ (Geom. 27).

Denote AC by b. Then $b=\sqrt{2^2-1^2}=\sqrt{3}$ (Geom. 344).

$\therefore\ \sin 30°=\frac{1}{2}$; $\cot 30°=\sqrt{3}$;

$\cos 30°=\dfrac{\sqrt{3}}{2}=\frac{1}{2}\sqrt{3}$; $\sec 30°=\dfrac{2}{\sqrt{3}}=\frac{2}{3}\sqrt{3}$;

$\tan 30°=\dfrac{1}{\sqrt{3}}=\frac{1}{3}\sqrt{3}$; $\csc 30°=2$.

Also $\sin 60^\circ = \frac{\sqrt{3}}{2} = \frac{1}{2}\sqrt{3}$;

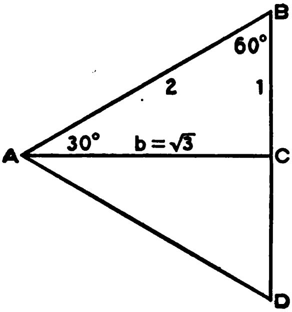
$\cos 60^\circ = \frac{1}{2}$;

$\tan 60^\circ = \sqrt{3}$;

$\cot 60^\circ = \frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3}$;

$\sec 60^\circ = 2$;

$\csc 60^\circ = \frac{2}{\sqrt{3}} = \frac{2}{3}\sqrt{3}$.

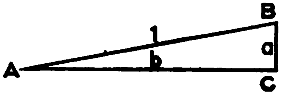


Note. A fraction having zero for its numerator and any quantity whatever for its denominator is equal to zero.

If the numerator of a fraction is any quantity whatever, and the value of the denominator becomes indefinitely small, the value of the fraction becomes indefinitely great. Inasmuch as it is impossible to divide by zero, it is generally understood that the value of a fraction whose denominator approaches zero as its limit may be made to exceed any number previously assigned, however great. Thus, as x becomes indefinitely small, the fraction $\frac{a}{x}$ becomes indefinitely great, provided a is constant. That is, as x is made less than any assigned quantity, $\frac{a}{x}$ may be made greater than any assigned quantity. We shall express this by saying that as x approaches zero, $\frac{a}{x}$ approaches ∞ (is read "infinity"). Then $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$ are equations which represent results of a limiting process, and are not to be taken by the student as usual equations, connecting numbers, because zero and infinity are not numbers, and the above equations, $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$, are used in a purely conventional way.

19. Problem. To find the values of the functions of 0° and of 90°.

Solution: Construct a right $\triangle ABC$, in which the hypotenuse $= 1$ and $\angle A$ is very small. Then $\angle B$ will be nearly 90° (Geom. 114). Denote BC by a and AC by b.



Now, $\sin A = a$; $\cos A = b$; $\tan A = \frac{a}{b}$.

Indefinitely diminish the $\angle A$ by bringing AB nearer and nearer into the position of the line AC. As a result,

$$\left.\begin{array}{l} \angle A \text{ will approach } 0^\circ; \\ a \text{ will approach } 0; \\ b \text{ will approach } 1; \\ \frac{a}{b} \text{ will approach } \frac{0}{b} \text{ or } 0. \\ \sin A \text{ will approach } \sin 0^\circ; \\ \cos A \text{ will approach } \cos 0^\circ; \\ \tan A \text{ will approach } \tan 0^\circ. \end{array}\right\}$$

Hence,

$$\left.\begin{array}{l} \mathbf{sin}\ 0^\circ = 0; \\ \mathbf{cos}\ 0^\circ = 1; \\ \mathbf{tan}\ 0^\circ = 0; \end{array}\right\} \text{(Geom. 242).}$$

$$\mathbf{cot}\ 0^\circ = \frac{1}{\tan 0^\circ} = \frac{1}{0} = \infty;$$

$$\mathbf{sec}\ 0^\circ = \frac{1}{\cos 0^\circ} = \frac{1}{1} = 1;$$

$$\mathbf{csc}\ 0^\circ = \frac{1}{\sin 0^\circ} = \frac{1}{0} = \infty.$$

$$\left.\begin{array}{ll} \mathbf{sin}\ 90^\circ = \cos 0^\circ = 1; & \mathbf{cot}\ 90^\circ = \tan 0^\circ = 0; \\ \mathbf{cos}\ 90^\circ = \sin 0^\circ = 0; & \mathbf{sec}\ 90^\circ = \csc 0^\circ = \infty; \\ \mathbf{tan}\ 90^\circ = \cot 0^\circ = \infty; & \mathbf{csc}\ 90^\circ = \sec 0^\circ = 1. \end{array}\right\} \text{(See § 13.)}$$

NOTE. Inasmuch as it is impossible to divide by zero, the expression $\cot 0^\circ = \frac{1}{0} = \infty$ is to be interpreted thus: as the angle decreases and approaches zero, the cotangent increases beyond any conceivable number. $\csc 0^\circ = \infty$; $\tan 90^\circ = \infty$; $\sec 90^\circ = \infty$ are to be interpreted similarly.

20. The values of these several functions of 0°, 30°, 45°, 60°, 90° are of such frequent use in trigonometry that they should be carefully memorized. For this purpose the following table will be found helpful:

	0°	30°	45°	60°	90°
sine	0	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$	1
cosine	1	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	0
tangent	0	$\frac{1}{3}\sqrt{3}$	1	$\sqrt{3}$	∞
cotangent	∞	$\sqrt{3}$	1	$\frac{1}{3}\sqrt{3}$	0
secant	1	$\frac{2}{3}\sqrt{3}$	$\sqrt{2}$	2	∞
cosecant	∞	2	$\sqrt{2}$	$\frac{2}{3}\sqrt{3}$	1

Note. In addition to the six trigonometric functions with which the student is already familiar, there are two functions that appear infrequently in elementary text-books. These are the **versedsine** and the **coversedsine.**

$$\text{vers } A = 1 - \cos A; \quad \text{covers } A = 1 - \sin A.$$

Example. Find the numerical value of:

$$\sin 45° \cos 90° + \cos^2 30° - \tan^2 30°.$$

Substituting,

$$\sin 45° \cos 90° + \cos^2 30° - \tan^2 30° = (\tfrac{1}{2}\sqrt{2})(0) + (\tfrac{1}{2}\sqrt{3})^2 - (\tfrac{1}{3}\sqrt{3})^2$$
$$= 0 + \tfrac{3}{4} - \tfrac{1}{3} = \tfrac{5}{12}. \quad \textit{Ans.}$$

EXERCISE 9

Find the numerical value of each:

1. $\sin 0° + \sin 90°$.
2. $\sin 30° - 2 \sin 90° + \cos 0°$.
3. $\cos 60° + \tan 45° - 3 \cos 90°$.
4. $\sin 60° - 2 \sin 0° + 3 \cos 0°$.
5. $\tan 30° - \sec 30° + 4 \cot 90°$.
6. $\cot 45° + 4 \sin 30° - 2 \cos 60°$.
7. $\csc 45° + 3 \sec 45° - \sec 0°$.
8. $\sin 60° + \cos 60° - 2 \tan 60°$.
9. $\sin 30° - \cos 30° + 2 \tan 30°$.
10. $2 \tan 45° - 5 \tan 0° - \sec 45°$.
11. $3 \sin 45° + \cos 45° - 3 \cot 60°$
12. $5 \sin 90° + 6 \cos 60° - 7 \tan 45°$.
13. $\sec 60° - \cos 60° + 4 \sin 60°$.
14. $\tan 45° + \cot 30° - \csc 45°$.
15. $2 \sin 45° - \cot 45° + 3 \cos 30°$.
16. $\sin 45° \cos 45° + \sin 30° \cos 30°$.
17. $\sin 45° \cos 30° + \cos 45° \sin 30°$.
18. $\cos 45° \cos 30° - \sin 45° \sin 30°$.

19. $\dfrac{\tan 45° + \tan 60°}{1 - \tan 45° \tan 60°}$.

20. $\dfrac{\tan 60° - \tan 30°}{1 + \tan 60° \tan 30°}$.

21. $1 - 2 \sin^2 30°$.

22. $2 \cos^2 60° - 1$.

23. $3 \cos 45° - 4 \cos^3 45°$.

24 $\sqrt{\dfrac{1 - \cos 60°}{2}}$.

25. $\dfrac{2 \tan 60°}{1 - \tan^2 60°}$.

26. $\sin^2 30° + \cos^2 45° - 2 \sin^2 60°$.

27. $\frac{1}{2} \sin^2 45° - \frac{2}{3} \tan^2 60° + 6 \tan^2 30°$.

28. $\sin^2 90° \cos^2 60° + \sec^2 0° \tan^2 60°$

29. $\sin 0° \cos 30° - \cos 90° \tan 45° + \sin^2 30° \cos^2 30°$.

30. $\tan^2 60° \cot^2 45° - \sec^2 0° \tan^2 0° + \tan 30° \cot 30° \sec 30°$.

31. $\sec 45° \tan 30° + \tan 60° \cot 90° - \sec 0° \csc 90° \sin 45°$.

Show that:

32. $\sin 60° = .8660^+$.

33. $\tan 60° = 1.7320^+$.

34. $\cos 45° = .7071^+$.

35. $\tan 30° = .5773^+$

36. $\sec 30° = 1.1546^+$.

37. $\csc 45° = 1.4142^+$.

38. If $x = 30°$, show that $\sin 2x = 2 \sin x \cos x$.

39. If $x = 30°$, show that $\cos 3x = 4 \cos^3 x - 3 \cos x$.

40. If $x = 30°$, show that $\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$.

41. If $x = 60°$, show that $\cos \frac{1}{2} x = \sqrt{\dfrac{1 + \cos x}{2}}$.

42. If $x = 30°$ and $y = 60°$, show that

$$\sin (x + y) = \sin x \cos y + \cos x \sin y.$$

43. If $A = 60°$ and $B = 30°$, show that

$$\tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}.$$

Find:

44. vers 0°; vers 30°; vers 45°; vers 60°; vers 90°.

45. covers 0°; covers 30°; covers 45°; covers 60°; covers 90°.

TRIGONOMETRIC EQUATIONS

21. A **trigonometric equation** is an equation involving one or more trigonometric functions.

The unknown quantity is primarily a trigonometric function, but ultimately it is an angle.

For example, in the equation $2 \sin x - 1 = 0$, the immediate unknown quantity is $\sin x$. Solving, $\sin x = \frac{1}{2}$. *Now* the unknown quantity is $\angle x$. But we know $\sin 30° = \frac{1}{2}$. $\therefore \angle x = 30°$.

The method of solving a trigonometric equation for an unknown angle is as follows:

I. *Reduce the given equation to an equation or equations, each containing only one trigonometric function.*

II. *Solve the resulting equation for this function.*

III. *From this value of the function determine the angle.*

These reductions are accomplished by use of the formulas in §§ 12, 13, and 15.

Ex. 1. Find θ if $2 \sin^2 \theta - \sin \theta = 0$.

By factoring, $\sin \theta (2 \sin \theta - 1) = 0$. $\therefore \sin \theta = 0$; $2 \sin \theta - 1 = 0$.

First, $\sin \theta = 0$. $\therefore \theta = 0°$.

Second, $\sin \theta = \frac{1}{2}$. $\therefore \theta = 30°$.

Verification. First, $2 \sin^2 0° - \sin 0° = 0 - 0 = 0$. Correct.
Second, $2 \sin^2 30° - \sin 30° = \frac{1}{2} - \frac{1}{2} = 0$. Correct.

Ex. 2. Solve $\sin^2 x - \cos^2 x = \frac{1}{2}$ for the value of x.

Substituting for $\cos^2 x$ its value $1 - \sin^2 x$,

$$\sin^2 x - (1 - \sin^2 x) = \tfrac{1}{2}. \quad \therefore 4 \sin^2 x - 3 = 0.$$

$\therefore \sin x = \frac{1}{2}\sqrt{3}$. Whence, $x = 60°$.

Verification. $\sin^2 60° - \cos^2 60° = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$. Correct.

Ex. 3. Solve $\sin A = \cos 3A$ for the value of A.

$\cos 3A = \sin(90° - 3A)$ (see § 13).

$\therefore$ substituting, $\sin A = \sin(90° - 3A)$.

$\therefore A = 90° - 3A$ and $A = 22° 30'$. VERIFICATION. (?).

Ex. 4. Find the value of y, if $\cot 7y = \tan(10° + y)$.

Substituting for $\cot 7y$ its value $\tan(90° - 7y)$,

$$\tan(90° - 7y) = \tan(10° + y).$$

$\therefore 90° - 7y = 10° + y$ and $8y = 80°$. $\therefore y = 10°$. VERIFICATION. (?)

EXERCISE 10

Solve for the unknown angle, and verify:

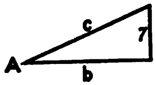
1. $\sin x = \frac{1}{2}$.
2. $\cos A = \frac{1}{2}\sqrt{2}$.
3. $\tan y = \sqrt{3}$.
4. $\sec A = 1$.
5. $\sin \theta = \frac{1}{2}\sqrt{3}$.
6. $\cot \theta = \frac{1}{3}\sqrt{3}$.
7. $\cos x = 0$.
8. $\csc x = \sqrt{2}$.
9. $3 \sec^2 x = 4$.
10. $\csc^2 x = 4$.
11. $\tan x = \infty$.
12. $\sin^2 x = 1$.
13. $\sin x(\sin x - 1) = 0$.
14. $(2 \cos x - 1)(\cos x - 1) = 0$.
15. $\tan x(2 \sin x - 1) = 0$.
16. $(\sec \theta - 2)(2 \sin^2 \theta - 1) = 0$.
17. $(\tan x - 1)(\tan^2 x - 3) = 0$.
18. $2 \sin^2 \theta = 3 \sin \theta - 1$.
19. $3 \cos \beta + 2 \sin^2 \beta = 3$.
20. $\tan^2 \theta = 3 \sec \theta - 3$.
21. $\cot^2 x + 3 = 3 \csc x$.
22. $2 \sin x \cos^2 x = \sin x$.
23. $2 \cos x \sin^2 x - \cos x = 0$.
24. $\sin 2A = \cos A$.
25. $\cos x = \sin 4x$.
26. $\tan 2y = \cot 3y$.
27. $\csc 5y = \sec 3y$.
28. $\sin(60° + x) = \cos 2x$.
29. $\cos \frac{1}{2}x = \sin 2x$.
30. $\tan(80° - \frac{1}{2}x) = \cot \frac{3}{8}x$.
31. $\sec(30° + \theta) = \csc 2\theta$.
32. $\cot(40° + \phi) = \tan \frac{1}{2}\phi$.
33. $\csc 3\theta = \sec(20° + \frac{1}{2}\theta)$.

CHAPTER II

SOLUTION OF RIGHT TRIANGLES

22. If one side* of a right triangle and any function of either acute angle are known, it is possible to find the length of the remaining sides of the triangle.

Ex. 1. If in $\triangle ABC$, $a = 7$ ft. and $\tan A = \frac{1}{3}$, find b and c.



$$\tan A = \frac{a}{b}. \quad \therefore b \cdot \tan A = a, \text{ or } b = \frac{a}{\tan A}.$$

$$\therefore b = \frac{7}{\frac{1}{3}} = 21 \text{ ft.} \quad \text{Also, } c = \sqrt{7^2 + 21^2}$$

$$= \sqrt{490} \qquad = 7\sqrt{10} \text{ ft.}$$

Ex. 2. If in $\triangle ABC$, $c = 10$ in. and $\tan A = \frac{1}{2}$, find a and b.

$$\sin A = \frac{a}{c}. \quad \therefore a = c \cdot \sin A. \quad \text{If } \tan A = \frac{1}{2}, \sin A = \frac{1}{\sqrt{5}} \text{ (see § 11).}$$

$$\therefore a = c \sin A = 10 \cdot \frac{1}{\sqrt{5}} = \frac{10}{\sqrt{5}} = 2\sqrt{5} \text{ in.}$$

$$b = \sqrt{10^2 - (2\sqrt{5})^2} = \sqrt{100 - 20} = \sqrt{80} = 4\sqrt{5} \text{ in.}$$

The student will notice that the function employed in each case is a function involving the given side.

EXERCISE 11

Find the remaining sides in each of the following right triangles:

1. $b = 24$ ft.; $\cos A = \frac{4}{5}$.
2. $c = 35$ in.; $\sin A = \frac{2}{5}$.
3. $a = 12.6$ yd.; $\tan A = \frac{3}{7}$.
4. $b = 5$ m.; $\sec A = 2$.

---

* In a right triangle the sides adjacent to the right angle may be called legs, arms, or sides.

5. $a = 24$ m.; $\cot A = 2$.

6. $c = 12.5$; $\tan A = \frac{3}{4}$.

7. $a = 240$; $\cos A = \frac{5}{13}$.

8. $b = 720$; $\csc A = 2.6$.

9. $c = 10$; $\cos A = \frac{1}{4}$.

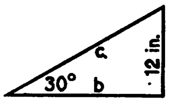
10. $b = 4$; $\sin A = \frac{2}{3}\sqrt{2}$.

23. We have derived and learned the actual numerical values of the functions of several angles, and if a right triangle contains one of these angles and a known side, the remaining sides may be determined.

Ex. 1. One of the acute angles of a right triangle is 30° and the side opposite is 12 in. Find the other sides.

$$\sin 30° = \frac{12}{c}. \quad \therefore c = \frac{12}{\sin 30°} = \frac{12}{\frac{1}{2}} = 24 \text{ in.}$$

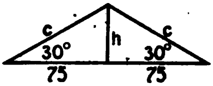
$$\tan 30° = \frac{12}{b}. \quad \therefore b = \frac{12}{\tan 30°} = \frac{12}{\frac{1}{3}\sqrt{3}} = \frac{36}{\sqrt{3}} = 12\sqrt{3} \text{ in.}$$



Ex. 2. The base of an isosceles triangle is 150 and the equal angles are each 30°. Find the equal sides and the altitude upon the base.

$$\cos 30° = \frac{75}{c}. \quad \therefore c = \frac{75}{\cos 30°} = \frac{75}{\frac{1}{2}\sqrt{3}} = 50\sqrt{3}.$$

$$\tan 30° = \frac{h}{75}. \quad \therefore h = 75 \cdot \tan 30°$$
$$= 75 \cdot \tfrac{1}{3}\sqrt{3} = 25\sqrt{3}.$$



The function employed in each case is a function involving the given line and the given angle.

EXERCISE 12

Find the remaining sides in each of the following right triangles:

1. $A = 60°$; $c = 100$ ft.

2. $A = 45°$; $b = 8$ in.

3. $B = 60°$; $c = 15$ in.

4. $A = 30°$; $c = 450$ m.

5. $A = 60°$; $a = 1.8$ yd.

6. $B = 30°$; $a = 30$ m.

7. The vertex angle of an isosceles triangle is 120° and the altitude from this vertex is 24. Find the sides of the triangle.

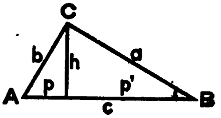
8. The sides of an equilateral triangle are each 20. Find the altitude.

9. The altitude of an equilateral triangle is 18. Find a side.

10. The hypotenuse of an isosceles right triangle is 6. Find the legs. Also find the altitude upon the hypotenuse.

11. In the adjoining figure if $\angle ACB = 90°$, $\angle A = 60°$ and $a = 10\sqrt{3}$, find c, b, p, p', h.

12. In the adjoining figure if $\angle ACB = 90°$, $\angle A = 60°$ and $p' = 300$, find a, b, c, p, h.



13. In the above figure if $\angle ACB = 90°$, $\angle B = 30°$, and $h = 24$, find a, b, c, p, p'.

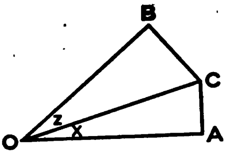


FIG. I.

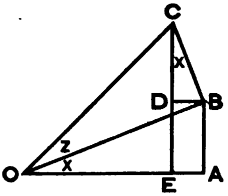


FIG. II.

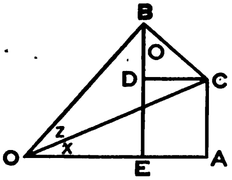


FIG. III.

In each of the above figures $\angle A$, $\angle OBC$, and $\angle BDC$ are right $\angle$s.

Denote the whole $\angle$ at O as $\angle O$, and its parts as $\angle x$ and $\angle z$.

Each line in the figures can be expressed in terms of some function of one of these angles and another line. Thus, in Fig. I $OB = OC \cdot \cos z$ or $= \dfrac{BC}{\tan z}$. Similarly express the following lines:

14. OB in Fig. II as belonging to two triangles.

15. OB in Fig. III.

16. BC in all figures.

17. OA in all figures.

18. OC in all figures.

19. BD in Figs. II and III.

20. CD in Figs. II and III.

Find the number of degrees in the angle A, of the following right triangles, in which the hypotenuses are represented by c:

21. $a = 10$; $c = 20$.

22. $a = 5$; $b = 5$.

23. $b = 7\sqrt{3}$; $c = 14$.

24. $a = 3$; $b = \sqrt{3}$.

25. $a = 3.6\sqrt{2}$; $c = 7.2$.

26. $c = .56$; $b = .28$.

SOLUTION OF RIGHT TRIANGLES BY NATURAL FUNCTIONS

24. The following table contains the numerical values of the sine, cosine, tangent, and cotangent of each angle containing an integral number of degrees, from 0° to 90°. These are not to be learned, but are given merely for reference.

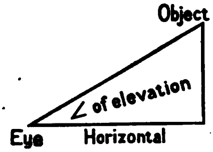
For functions of angles not greater than 45° use the titles at the tops of the columns and the left-hand column of degrees. Thus, cos 18° = .9511; tan 32° = .6249. For functions of angles greater than 45° use the titles at the bottoms of the columns and the right-hand column of degrees. Thus, sin 53° = .7986; cot 66° = .4452; cos 80° = .1736.

Table of Sines, Cosines, Tangents, Cotangents

	Sin	Cos	Tan	Cot			Sin	Cos	Tan	Cot	
0°	.0000	1.0000	.0000	Infinity	**90°**	**23°**	.3907	.9205	.4245	2.3559	**67°**
1°	.0175	.9998	.0175	57.29	**89°**	**24°**	.4067	.9135	.4452	2.2460	**66°**
2°	.0349	.9994	.0349	28.64	**88°**	**25°**	.4226	.9063	.4663	2.1445	**65°**
3°	.0523	.9986	.0524	19.08	**87°**	**26°**	.4384	.8988	.4877	2.0503	**64°**
4°	.0698	.9976	.0699	14.30	**86°**	**27°**	.4540	.8910	.5095	1.9626	**63°**
5°	.0872	.9962	.0875	11.43	**85°**	**28°**	.4695	.8829	.5317	1.8807	**62°**
6°	.1045	.9945	.1051	9.5144	**84°**	**29°**	.4848	.8746	.5543	1.8040	**61°**
7°	.1219	.9925	.1228	8.1443	**83°**	**30°**	.5000	.8660	.5774	1.7321	**60°**
8°	.1392	.9903	.1405	7.1154	**82°**	**31°**	.5150	.8572	.6009	1.6643	**59°**
9°	.1564	.9877	.1584	6.3138	**81°**	**32°**	.5299	.8480	.6249	1.6003	**58°**
10°	.1736	.9848	.1763	5.6713	**80°**	**33°**	.5446	.8387	.6494	1.5399	**57°**
11°	.1908	.9816	.1944	5.1446	**79°**	**34°**	.5592	.8290	.6745	1.4826	**56°**
12°	.2079	.9781	.2126	4.7046	**78°**	**35°**	.5736	.8192	.7002	1.4281	**55°**
13°	.2250	.9744	.2309	4.3315	**77°**	**36°**	.5878	.8090	.7265	1.3764	**54°**
14°	.2419	.9703	.2493	4.0108	**76°**	**37°**	.6018	.7986	.7536	1.3270	**53°**
15°	.2588	.9659	.2679	3.7321	**75°**	**38°**	.6157	.7880	.7813	1.2799	**52°**
16°	.2756	.9613	.2867	3.4874	**74°**	**39°**	.6293	7771	.8098	1.2349	**51°**
17°	.2924	.9563	.3057	3.2709	**73°**	**40°**	.6428	.7660	.8391	1.1918	**50°**
18°	.3090	.9511	.3249	3.0777	**72°**	**41°**	.6561	.7547	.8693	1.1504	**49°**
19°	.3256	.9455	.3443	2.9042	**71°**	**42°**	.6691	.7431	.9004	1.1106	**48°**
20°	.3420	.9397	.3640	2.7475	**70°**	**43°**	.6820	.7314	.9325	1.0724	**47°**
21°	.3584	.9336	.3839	2.6051	**69°**	**44°**	.6947	.7193	.9657	1.0355	**46°**
22°	.3746	.9272	.4040	2.4751	**68°**	**45°**	.7071	.7071	1.0000	1.0000	**45°**
	Cos	Sin	Cot	Tan			Cos	Sin	Cot	Tan	

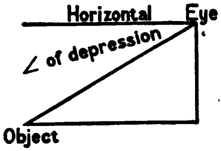
The student has already derived several of the values given in the preceding table, but the method of determining the most of them is laborious and quite beyond the attainment of the pupil at this point in the study of Trigonometry.

25. The **angle of elevation** of an object is the angle formed by a horizontal plane through the eye of the observer and a line from the observer's eye upward to the object.



Thus, the angle of elevation of a balloon is the angle between the horizontal plane at the person observing the balloon and the line from the observer's eye upward to the balloon.

The **angle of depression** of an object is the angle formed by a horizontal plane through the eye of the observer and a line from the observer's eye downward to the object.

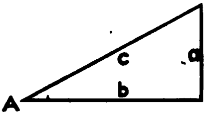


Thus, the angle of depression of a boat observed from the top of a cliff is the angle between the horizontal plane at the person observing the boat and the line from the observer's eye downward to the boat.

The instrument used by surveyors to measure angles in a vertical plane and angles in a horizontal plane is the **transit.** The two levels at the base enable the operator to secure a horizontal plane, and the level attached to the telescope insures a horizontal line of vision. There are two graduated scales for measuring angles: one at the base, used for angles in a horizontal plane; the other on the axle of the telescope, for measuring angles in a vertical plane. (See frontispiece.)

26. By the use of the table on page 35 we are able to find the remaining sides of any right triangle one of whose sides is known, provided the known acute angle is an integral number of degrees.

Ex. 1. If in a right triangle $\angle A = 25°$ and $a = 20$ in., find b and c.



$$\tan A = \frac{a}{b}. \quad \therefore b = \frac{a}{\tan A} = \frac{20}{.4663} = 42.89 \text{ in.}$$

$$\sin A = \frac{a}{c}. \quad \therefore c = \frac{a}{\sin A} = \frac{20}{.4226} = 47.32 \text{ in.}$$

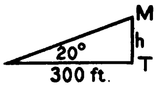
The function selected must contain the given side.

Ex. 2. If in a right triangle $\angle A = 55°$ and $c = 15$ in., find a and b.

$$\sin A = \frac{a}{c}. \quad \therefore a = c \cdot \sin A = 15 \times .8192 = 12.29 \text{ in.}$$

$$\cos A = \frac{b}{c}. \quad \therefore b = c \cdot \cos A = 15 \times .5736 = 8.6 \text{ in.}$$

Ex. 3. Standing 300 ft. from a monument and in the same horizontal plane as its base, I observe the angle of elevation of the top of the monument is 20°. Find its height.



$$\tan 20° = \frac{h}{300}. \quad \therefore h = 300 \cdot \tan 20°$$
$$= 300 \times .3640 = 109.2 \text{ ft.}$$

EXERCISE 13

1. In a right triangle whose hypotenuse is 100 ft. an acute angle is 40°. Find the lengths of the legs.

2. One leg of a right triangle is 20 in. and the opposite angle is 70°. Find the other sides.

3. One leg of a right triangle is 20 in. and the adjoining acute angle is 70°. Find the other sides.

4. The hypotenuse of a right triangle is 10.5 and one acute angle is 33°. Find the other sides.

5. At a point 320 ft. away and on a level with the base of a tower the angle of elevation of the top of the tower is observed to be 28°. What is the height of the tower?

6. On the opposite shores of a lake are two posts A and B. Perpendicular to the line AB, along one shore, a line $BC = 300$ yd. is measured. The $\angle ACB$ is 50°. Find the width of the lake, that is, the length of the line AB.

7. The angle of ascent of a road up a hill is 10°. If one walks a mile up the road, how many feet has he risen?

8. Each of the equal angles of an isosceles triangle is 65° and the equal sides are each 25 ft. Find the lengths of the third side and of the altitude upon it.

9. The base of an isosceles triangle is 24 and the angle at each end of the base is 72°. Find the equal sides and the altitude upon the given base.

10. The base of an isosceles triangle is 120 and the angle opposite is 100°. Find the equal sides and the altitude upon the given base.

11. The altitude of an isosceles triangle is 10 and the angle it bisects is 84°. Find the sides of the triangle.

12. The altitude of an isosceles triangle is 32 and each of the equal angles is 32°. Find the sides of the triangle.

13. The angle of depression of a buoy one mile from the foot of a cliff, observed from the top of the cliff, is 10°. Find the height of the cliff in feet.

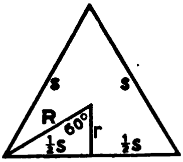
14. The angle of elevation of the topmost point of the Philadelphia City Hall, measured along a level street and 990 ft. from the base of the building, is 29°. Find the extreme height of the City Hall.

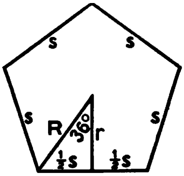
15. From the roof of a building on one side of a street 50 ft. wide, the angle of depression of the other side of the street is 79°. Find the height of the building.

16. From an office window the angle of depression of an object 500 ft. from the building, measured along a level street, is 15°. How high is the window above the street?

17. Each side of an equilateral triangle is 8. Find the radii of the inscribed and circumscribed circles.

18. The apothem of an equilateral triangle is 10. Find the side of the triangle.

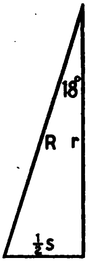




19. In a regular pentagon whose side is 12 find R and r.

20. In a regular pentagon whose apothem is 8 find R and s.

21. In a regular hexagon whose apothem is 6 find R and s.



22. In a regular decagon whose side is 16 find R and r. (See adjoining figure.)

23. In a regular decagon whose apothem is 4 find R and s.

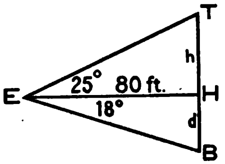
24. In a regular dodecagon whose side is 10 find R and r.

25. In a regular dodecagon whose radius is 5 find r and s.

26. Find R and s in a regular dodecagon whose apothem is 3.

27. Find r and s in a regular decagon whose radius is 15.

28. The horizontal distance from my window to a flagpole is 80 ft. The angle of elevation of the top of the pole is 25° and the angle of depression of the foot of the pole is 18°. Find the length of the pole above ground.



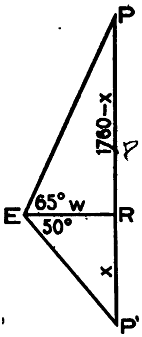
29. From a point on one bank of a river I can see two posts along the opposite bank and one mile apart, one upstream, the other down. The line directly across the river at the point from which I am observing makes an angle of 65° with the line drawn from this point to one post, and 50° with the line drawn to the other post. Find the width of the river in yards.

SUGGESTION. In $\triangle ERP'$ $x = w \cdot \tan 50°$;

in $\triangle ERP$ $x = 1760 - w \cdot \tan 65°$.

$$\therefore w \cdot \tan 50° = 1760 - w \cdot \tan 65° \text{ (Ax. 1)}.$$

$$\therefore w = \frac{1760}{\tan 50° + \tan 65°} = \text{etc.}$$



30. Solve Problem 29 if P and P' are on the same side of ER.

31. The height of a certain tower is 300 ft. From the top of another tower the angle of elevation of the top of the first is 12° and the angle of depression of its base is 53°. Find the distance between the towers.

32. Solve Problem 31, substituting the word "depression" for "elevation."

33. The chimney of a factory extends 50 ft. above the roof of the factory. At a point on the level with the base of the factory the angles of elevation of the top of the chimney and of the roof are known to be 73° and 56° respectively. Find the height of the factory.

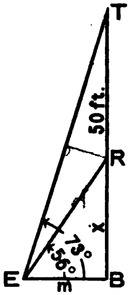
SOLUTION. Denote the distance from the factory to the observer by m and the height of the factory by x.

In $\triangle EBR$ $m = \dfrac{x}{\tan 56^\circ}$.

In $\triangle EBT$ $m = \dfrac{x+50}{\tan 73^\circ}$. $\therefore \dfrac{x}{\tan 56^\circ} = \dfrac{x+50}{\tan 73^\circ}$.

$$\text{Solving, } x = \frac{50 \tan 56^\circ}{\tan 73^\circ - \tan 56^\circ} = \frac{50 \times 1.4826}{3.2709 - 1.4826}$$

$$= \frac{74.13}{1.7883} = 41.45^{+}\text{ ft.}$$



34. Upon a hill stands a tower 200 ft. high. At a point in the horizontal plane of the foot of the hill the angles of elevation of the top and bottom of the tower are observed to be 51° and 37° respectively. Find the height of the hill.

35. From the top of a tower two milestones are observed in a straight horizontal road leading to the foot of the tower. Their angles of depression are 3° and 27°. Find the height of the tower in feet.

36. Upon a house stands a flagstaff whose length is a ft. From a point in a horizontal street leading to the house, the angles of elevation of the top and bottom of the staff are γ° and δ° respectively. Find a formula from which the height of the house could be determined if a, γ, and δ were known. Also find a formula for the distance from the point of observation to the house.

37. Two observers 650 ft. apart, facing each other, in a horizontal plane, observe at the same time, the angles of elevation of a balloon in their vertical plane, to be 50° and 73°. What is the height of the balloon above the ground?

38. Two observers d ft. apart, facing each other, in a horizontal plane, observe at the same time, the angles of elevation of a balloon in their vertical plane, to be θ° and ϕ°. Derive a formula from which the height of the balloon can be determined, if d, θ, and ϕ are known.

39. Walking along a straight level road leading to the foot of a hill, I observed the angle of elevation of the top of the hill to be α°. After proceeding m yards nearer the hill the angle of observation was β°. Derive a formula for the height of the hill in terms of m, α, and β. $X = \frac{m \sin\alpha \sin\beta}{\sin(\beta - \alpha}$

SOLUTION OF RIGHT TRIANGLES BY LOGARITHMS

27. The complete solution of triangles usually involves much multiplication and division, as has become apparent from the preceding exercise. Logarithms render the greatest economies of time and labor in this sort of computation, that have ever been devised by man.

There are many really excellent sets of tables of logarithms printed, and these are usually supplied with sufficient explanation and illustration to enable the average student to become skillful in their operation without difficulty or unreasonable delay. In this book the treatment of logarithms is given in Chapter VI, and several exercises are supplied to aid in rendering the learner familiar with the handling of the tables. This familiarity is essential at this juncture in the study of Trigonometry, and before further progress can be made here, the class should at least acquire the ability to solve the examples of Exercises 52, 53, 57, 58.

28. A right triangle is **determined** if any two of its five parts are known, one of these two being a side. In other words, there can be only one distinct right triangle having a given leg and a given hypotenuse; or a given side and a given acute angle. There are always three parts to be found. The **solution** of a right triangle is accomplished if these three unknown parts have been discovered.

29. The **definition** of each trigonometric function is an equation containing three of the five elements of the triangle.

If any two of these three are known, the solution of this equation will supply the third. Hence, in the solution of a right triangle,

I. Select that function of one of the angles whose definition gives an equation containing two known quantities and one unknown quantity. (Avoid secant and cosecant.)

II. Solve the equation for this unknown quantity.

III. By a second equation not involving the part already found* find another unknown.†

IV. Always write the logarithmic scheme of the solution fully before looking up a single logarithm.

Ex. 1. In a right △, if $c = 75.46$ and $A = 38° 48' 28''$, find $\angle B$, a, b. Here $\angle B = 90° - \angle A = 51° 11' 32''$.

The function containing c, A, a is $\sin A = \frac{a}{c}$. $\therefore a = c \cdot \sin A$.

The function containing c, A, b is $\cos A = \frac{b}{c}$. $\therefore b = c \ \cos A$.

Scheme for finding a	Logarithms	Scheme for finding b	Logarithms
$\log c$	$= 1.87772$	$\log c$	$= 1.87772$
$\log \sin A$	$= 9.79706 - 10$	$\log \cos A$	$= 9.89168 - 10$
$\log a$	$= 1.67478$	$\log b$	$= 1.76940$
	$\therefore a = 47.291.$		$\therefore b = 58.803.$

Verification. The equation connecting the parts found is called the "verifying formula," or the "check formula." In this case the check is $\tan B = \frac{b}{a}$. $\therefore$ $\log \tan B = \log b + \operatorname{colog} a$, if the results are correct.

$$\log \tan B = .09461; \ \log b + \operatorname{colog} a = .09462.$$

Inasmuch as logarithmic computation is only a method of very close approximation, these results are as nearly identical as we can expect them to be, even though all our values were known to be correct.

---

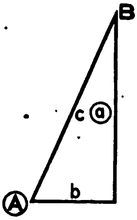
* Because if the part already found is not correct, the part found by using it will not be correct.

† The equation $c = \sqrt{a^2 + b^2}$ for finding c, when a and b are given, is not adapted to logarithmic computation. It is better, therefore, to find $\angle A$ first, be careful to secure correctness, and use $\angle A$ in finding c.

$\frac{\text{[illegible]}}{\text{[illegible]}}$ = $\frac{60''}{X}$. [illegible] $\frac{10}{X}$

Ex. 2. If $a = .8347$ and $A = 66° 26' 16''$, find $\angle B$, b, c.

Here $\angle B = 90° - \angle A = 23° 33' 44''$.



$\tan A = \frac{a}{b}$. $\therefore b = \frac{a}{\tan A}$.

$\log a = 9.92153 - 10$

$\text{colog} \tan A = 9.63959 - 10$

$\log b = 9.56112 - 10$

$\therefore b = .364017$

$\sin A = \frac{a}{c}$. $\therefore c = \frac{a}{\sin A}$.

$\log a = 9.92153 - 10$

$\text{colog} \sin A = 0.03781$

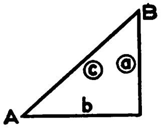
$\log c = 9.95934 - 10$

$\therefore c = .91062$

Verification. $\sin B = \frac{b}{c}$ is the check. $\therefore \log \sin B = \log b + \text{colog}\, c$.

$\log \sin B = 9.60178 - 10$; $\log b + \text{colog}\, c = 9.60178 - 10$. Correct.

Ex. 3. If $c = 138.7$ and $a = 91.8$, find $\angle A$, $\angle B$, b.



$\sin A = \frac{a}{c}$.

$\log a = 1.96284$

$\text{colog}\, c = 7.85792 - 10$

$\log \sin A = 9.82076 - 10$

$\therefore \angle A = 41° 26' 28''$

and $\angle B = 48° 33' 32''$

If A is incorrect, B cannot be correct.

The equation joining a, c, b, is

$$b = \sqrt{c^2 - a^2} = \sqrt{(c+a)(c-a)}.$$

$c + a = 230.5$;

$c - a = 46.9$

$\log (c + a) = 2.36267$

$\log (c - a) = 1.67117$

$2\overline{)4.03384}$

$\log b = 2.01692$

$\therefore b = 103.973.$

Verification. $\tan A = \frac{a}{b}$. $\therefore \log \tan A = \log a + \text{colog}\, b$.

$\log \tan A = 9.94591 - 10$; $\log a + \text{colog}\, b = 9.94592 - 10$. Correct.

EXERCISE 14

Find, by logarithms, the remaining parts of the following right triangles:

1. $a = 78$; $A = 62° 10'$.
2. $a = 10$; $B = 67°$.
3. $c = 627$; $A = 23° 30'$.
4. $a = 35$; $c = 62$.
5. $a = 27.3$; $b = 41.8$.
6. $b = .7548$; $A = 66° 51'$.
7. $c = 18.75$; $B = 55° 32'$.
8. $a = 17.5$; $c = 24.4$.
9. $a = 407.5$; $A = 63° 10' 30''$.
10. $b = .0876$; $c = .1458$.

11. $a=3255$; $b=4877$.

12. $b=.7174$; $A=38^\circ 40' 45''$.

13. $c=1.7163$; $B=49^\circ 19' 50''$.

14. $a=9.4146$; $b=11.716$.

15. $b=3.86$; $c=9.75$.

16. $a=2.5681$; $A=27^\circ 40' 15''$.

17. $b=93.217$; $A=76^\circ 14' 38''$.

18. $a=72.913$; $b=38.411$.

19. $A=61^\circ 14' 17''$; $c=321.76$.

20. $c=.8532$; $B=48^\circ 18' 14''$.

21. $c=.07165$; $a=.029142$.

22. $b=8.1175$; $c=11.3907$.

23. $A=49^\circ 49' 27''$; $a=1567.81$.

24. $a=7.10042$; $b=5.20093$.

25. A line 7 in. long is inclined at an angle of 33° 18′ 40″ to another line. What is the length of the projection of the first line upon the second?

26. At a distance of 226.3 ft. from the foot of a monument, the angle of elevation of its top is 21° 30′ 18″. Find the height of the monument.

27. Along one bank of a river a straight line AB is measured and found to be 130 ft. Opposite the point B is a tree T, and the angle TAB is found to be 58° 40′ 20″. Find the width of the river; that is, the length of BT.

28. From the roof of a house 40 ft. high the angle of elevation of the top of a chimney across a level street is found to be 40° 25′, and the angle of depression of the base of the chimney is 35° 20′. Find the height of the chimney.

29. The shadow of a vertical post 20 ft. high is 16.5 ft. Find the angle of elevation of the sun.

30. From the top of a lighthouse 125 ft. above the sea level the angles of depression of two buoys, in the same vertical plane with the lighthouse, were 15° 24′ and 70° 40′. Find the distance between the buoys.

31. If the hypotenuse of a right triangle is 18.054 and one leg is 10.586, find the other leg and an acute angle.

32. If the hypotenuse of a right triangle is .763 and one leg is .586, find the other leg and an acute angle.

EXERCISE 15

In the following isosceles triangles $\angle A = \angle B$ and $a = b$; the altitude upon the base $c = h$ and the vertex angle is C. Find the remaining parts in each.

1. $A = 47° 28'$, $a = 275.4$.

2. $C = 80° 25' 40''$, $b = .8764$.

3. $B = 67° 28' 45''$, $c = 1.4876$.

4. $a = 19.516$, $c = 23.578$.

5. $c = 4.8915$, $A = 70° 20' 36''$.

6. $b = .008541$, $B = 63° 27' 42''$.

7. $A = 50° 30' 25''$, $h = .09146$.

8. $a = 931.45$, $h = 783.87$.

9. A wedge measures 3 in. across the base and 15 in. along the side. Find the angle at its vertex.

10. The legs of a pair of dividers are 5 in. long and are set at an angle of 50° 40′. What is the distance between their points?

In the following regular polygons, n = number of sides, C = central angle, A = an angle of the polygon, r = apothem, s = a side, R = radius of circumscribed circle

Find the remaining parts:

11. $n = 5$, $R = 75$.

12. $n = 5$, $r = 75$.

13 $n = 8$, $s = 75$.

14. $C = 30°$, $R = 38.76$

15. $A = 160°$, $R = 7.8$.

16. $n = 20$, $r = 5$.

17 $n = 16$, $s = 50$.

18. $n = 7$, $r = 100$.

30. The **area** of a triangle is equal to half the product of its base and altitude (Geom. 378). Let us denote the area by the letter S and give the usual significance to the other letters.

Then, for a right △, $S = \frac{1}{2} a \cdot b$.

$$\therefore \log S = \text{colog } 2 + \log a + \log b.$$

For an isosceles △, $S = \frac{1}{2} c \cdot h$.

$$\therefore \log S = \text{colog } 2 + \log c + \log h.$$

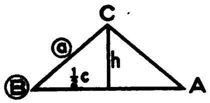
For a regular polygon $S = n\left(\frac{1}{2} s \cdot r\right)$.

$$\therefore \log S = \log n + \text{colog } 2 + \log s + \log r.$$

Ex. Find the area of an isosceles △, each of whose equal angles is 40° 27′ 15″ and whose equal sides are each 3.756.

Here $S = \frac{1}{2} c \cdot h$.

$\therefore \log S = \text{colog } 2 + \log c + \log h$.



$\sin B = \frac{h}{a}$. $\therefore h = a \cdot \sin B$.

$$\begin{aligned} \log a &= 0.57473 \\ \log \sin B &= \underline{9.81214 - 10} \\ \log h &= .38687 \end{aligned}$$

$\cos B = \frac{\frac{1}{2}c}{a}$.

$\therefore c = 2\, a \cos B$.

$$\begin{aligned} \log 2 &= 0.30103 \\ \log a &= 0.57473 \\ \log \cos B &= \underline{9.88134 - 10} \\ \log c &= .75710 \end{aligned}$$

$$\begin{aligned} \therefore \text{colog } 2 &= 9.69897 - 10 \\ \log c &= .75710 \\ \log h &= \underline{.38687} \\ \log S &= .84294 \\ \therefore S &= 6.9653, \text{ area.} \end{aligned}$$

EXERCISE 16

Find the areas of the following right triangles:

1. $A = 73° 42′$, $a = 6.43$.
2. $a = .7168$, $b = .9437$.
3. $a = 1.9114$, $c = 2.7165$.
4. $c = 378.54$, $A = 28° 48′ 38″$.
5. $b = 9.417$, $c = 20.716$.
6. $b = .008765$, $A = 69° 27′ 28″$

Find the areas of the following isosceles triangles:

7. $A = 50° 40′$, $a = 8.41$.
8. $a = 20.148$, $c = 28.417$.
9. $a = 5.417$, $C = 82° 7′ 14″$.
10. $c = .8766$, $C = 69° 43′ 30″$

Find the areas of the following regular polygons:

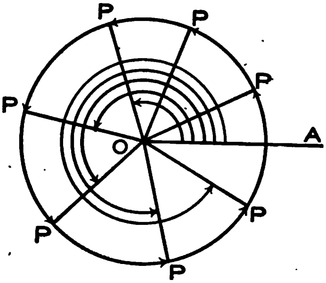
11. $n = 5$, $R = 20$.
12. $n = 8$, $r = 20$.
13. $n = 10$, $s = 30$.
14. $n = 12$, $R = 75$.
15. $n = 32$, $s = 32$.
16. $n = 7$, $r = 70$.

CHAPTER III

THE TRIGONOMETRIC FUNCTIONS OF ANY ANGLE

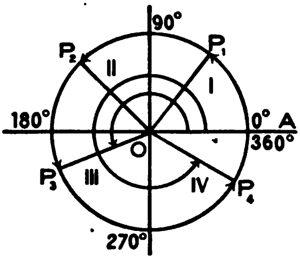
31. Thus far in our study we have been concerned with the angles of a right triangle, but in order that we may be prepared to solve oblique triangles, it is necessary that we extend our treatment of the functions of acute angles to those of any angle. In order to do this most intelligently it is desirable to conceive an angle as generated by a line revolving about one of its points, and always remaining in the same plane.

The line *OP* revolving in a plane, about a point *O*, from an initial position, *OA*, will generate about the point *O*, angles of all possible magnitudes. At first ∠ *AOP* is acute; then ∠ *AOP* becomes a right angle; then it is obtuse; later ∠ *AOP* is 180°. After this, in the revolution, ∠ *AOP* is reflex, or reëntrant, being greater than 180°. Then it becomes 270° and 360° and may commence a second revolution and continue revolving indefinitely. We will, for the most part, confine ourselves to angles not exceeding 360°.



The revolving line, or the generating line, is called the **terminal line.** The original position, from which the revolution commences, is called the **initial line.** These lines are the **sides** of the angle. The point about which the line rotates is the **vertex** of the angle.

32. If a line is drawn through the vertex of the angle perpendicular to the initial line, this line and the initial line



(extended) divide the plane into four regions called **quadrants**, which are numbered in order, First, Second, Third, and Fourth, or I, II, III, and IV.

The position of the terminal line denotes the quadrant in which the angle is said to be.

Any acute angle is in the first quadrant, $\angle AOP_1$.

Any obtuse angle is in the second quadrant, $\angle AOP_2$.

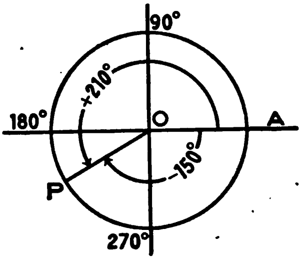
Any angle between 180° and 270° is in the third quadrant, $\angle AOP_3$.

Any angle between 270° and 360° is in the fourth, $\angle AOP_4$.

Any angle between 360° and 450° is in the first, etc.

33. Positive and negative angles. Angles are regarded as **positive** if the revolution of the terminal line has been in the opposite direction of the motion of the hands of a clock. This motion is called **counter-clockwise**.

Angles are regarded as **negative** if the revolution of the terminal line has been in the same direction as the motion of the hands of a clock.



The $\angle AOP$ if generated positively, $= +210°$, but if generated negatively, $= -150°$. It is evident that for every position of the terminal line there are two angles, one positive, the other negative, each less than 360° and their sum (regardless of sign) is 360°.

Hence, to find the positive angle, corresponding to a given negative angle, we understand 360 written before the minus sign.

Ex. 1. $-80°$ corresponds to $360° - 80° = +280°$.

Ex. 2. $\theta - 200°$ corresponds to $360° - 200° + \theta = 160° + \theta$.

Ex. 3. $-650°$ corresponds to $720° - 650° = +70°$.

EXERCISE 17

Find the positive angle corresponding to each of the following negative angles (each letter represents an acute angle):

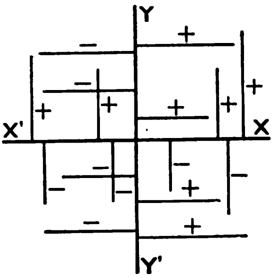
1. $-100°$.
2. $-200°$.
3. $-330°$.
4. $-75°$.
5. $-10°$.
6. $-A$.
7. $x-180°$.
8. $\theta-90°$.
9. $A-270°$.
10. $\phi-300°$.

Draw a figure showing in which quadrant each of the following angles lies:

11. $80°$. 1
12. $220°$. 3
13. $300°$.
14. $240°$.
15. $350°$.
16. $150°$.
17. $-20°$.
18. $-300°$. 1
19. $-210°$.
20. $-120°$.
21. $-170°$. 3
22. $270°-x$.
23. $90°+\theta$.
24. $180°+A$.
25. $270°+x$.
26. $390°$. 1
27. $500°$. 2
28. $400°+x$.
29. $1000°$. 4
30. $-540°$.
31. $-510°$. 2
32. $540°-\theta$. 2
33. $-1000°$. 1
34. $x-270°$.
35. $\theta-750°$.
36. $y-630°$.
37. $x-100°$.
38. $x-1000°$.

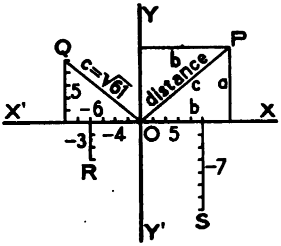
34. Positive and negative lines. Lines are regarded as positive or negative according to their directions from a line or point of reference. If a plane is divided into two regions by a straight line, all straight lines perpendicular to this given line and on one side of it are **positive** and those perpendicular to it but on the other side of it are **negative.**

Thus, if XX' and YY' are drawn at right angles, every line ⊥ to XX' and above it is regarded positive; every line ⊥ to XX' and below it is regarded negative. Also, every line ⊥ to YY' and on the right of it is regarded positive; every line ⊥ to YY' and on the left of it is regarded negative.



The lines of reference, XX' and YY', are called **axes.**

35. Rectangular coördinates of a point. The position of a point in a plane is determined if its perpendicular distance from each of two perpendicular axes is known, together with the signs of these distances.



The perpendicular distance from the point to the YY' axis is called the **abscissa** of the point. The perpendicular distance from the point to the XX' axis is called the **ordinate** of the point. The line from the intersection of the axes to the point is called the **distance.**

Thus, of point P, b is the abscissa, a is the ordinate, c is the distance.

Of point Q, -6 is the abscissa, 5 is the ordinate, $\sqrt{61}$ is the distance.

Of point R, -4 is the abscissa and -3 is the ordinate.

Of point S, 5 is the abscissa and -7 is the ordinate.

From these illustrations it is seen that the abscissa and the ordinate may be either positive or negative according to the position of the point. Briefly,

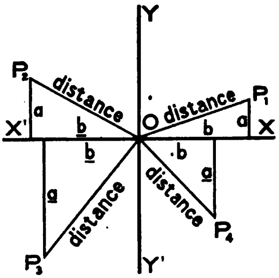
For points in quadrant I, both b and a are positive.

For points in quadrant II, b is negative and a is positive.

For points in quadrant III, both b and a are negative.

For points in quadrant IV, b is positive and a is negative

The **distance** is considered always **positive.** Hereafter, in the diagrams, symbols representing negative lines are underscored.



36. It is apparent, therefore, that the trigonometric functions of an angle may be defined in terms of *abscissa*, *ordinate*, and *distance.* If one side of an angle is considered the XX' axis and a point is taken in the other side of the angle,

The **sine** of an angle, $\frac{a}{c} = \frac{\textbf{ordinate}}{\textbf{distance}}$.

The **cosine** of an angle, $\frac{b}{c} = \frac{\textbf{abscissa}}{\textbf{distance}}$.

The **tangent** of an angle, $\frac{a}{b} = \frac{\textbf{ordinate}}{\textbf{abscissa}}$.

The **cotangent** of an angle, $\frac{b}{a} = \frac{\textbf{abscissa}}{\textbf{ordinate}}$.

The **secant** of an angle, $\frac{c}{b} = \frac{\textbf{distance}}{\textbf{abscissa}}$.

The **cosecant** of an angle, $\frac{c}{a} = \frac{\textbf{distance}}{\textbf{ordinate}}$.

37. Signs of the functions of any angle.

I. Of any angle in the first quadrant, all the functions are positive.

II. Of any angle in the second quadrant, those functions involving b are negative. The others are positive. The sine and cosecant only are positive.

III. Of any angle in the third quadrant, those functions involving either a or b (but not both) are negative. The others are positive. The tangent and cotangent only are positive.

IV. Of any angle in the fourth quadrant, those functions involving a are negative. The others are positive. The cosine and secant only are positive.

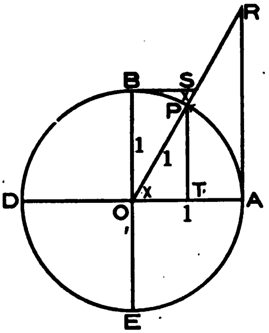
EXERCISE 18

Name the quadrants in which

1. The sine is positive. 12
2. The cosine is positive. 1 4
3. The tangent is positive.
4. The cotangent is positive.
5. The secant is positive.
6. The cosecant is positive.

7. The sine is negative. 34
8. The cosine is negative. 23
9. The tangent is negative. 24
10. The cotangent is negative. 24
11. The secant is negative. 2
12. The cosecant is negative. 34

38. Geometrical method of representing the functions of an angle. A **unit circle** is a circle whose radius is a unit in length. Draw a unit circle having the vertex of the given angle as center and draw the initial diameter *AD* and the diameter *BE* ⊥ to *AD*. Denote the given ∠ *AOP* by x. Draw *PT* ⊥ to *AD*, *AR* ⊥ to *AD* at *A*, meeting *OP* (produced) at *R*, and *BS* ⊥ to *BE* meeting *OP* (produced) at *S*. ∠ *BSO* = ∠x (Geom. 97). We now have three right ▲ *OPT*, *OAR*, and *OBS*, each containing an angle = ∠x and each containing one side = 1.



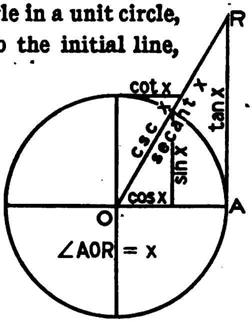
These ▲ are similar (Geom. 315).

$$\text{In } \triangle\, OPT,\ \begin{cases} \sin x = \dfrac{PT}{1} = PT, \\[2ex] \cos x = \dfrac{OT}{1} = OT. \end{cases}$$

$$\text{In } \triangle\, OAR,\ \begin{cases} \tan x = \dfrac{AR}{1} = AR, \\[2ex] \sec x = \dfrac{OR}{1} = OR. \end{cases}$$

$$\text{In } \triangle\, OBS,\ \begin{cases} \cot x = \dfrac{BS}{1} = BS, \\[2ex] \csc x = \dfrac{OS}{1} = OS. \end{cases}$$

Let us call *OA* the initial radius, *OB* the vertical radius, *OP* the terminal radius. *AR* is tangent to ⊙ *O* (Geom. 215).



Then, **provided the given angle is a central angle in a unit circle,**

The *sine* of an angle is the perpendicular to the initial line, from the extremity of the terminal radius.

The *cosine* of an angle is the distance from the vertex to the foot of the sine.

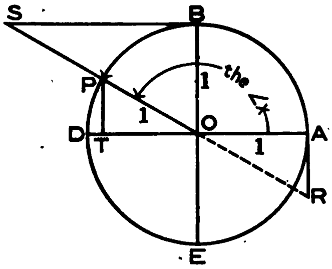
The *tangent* of an angle is the length of the tangent to the circle (drawn at the extremity of the initial radius) from this extremity to the point at which this tangent line meets the terminal line.

The *cotangent* of an angle is the length of the tangent to the circle (drawn at the extremity of the vertical radius) from this extremity to the point at which this tangent line meets the terminal line.

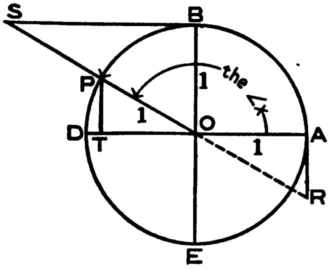
The *secant* of an angle is the distance along the terminal line from the center to the point at which it meets the tangent to the circle drawn at the extremity of the initial radius.

The *cosecant* of an angle is the distance along the terminal line from the center to the point at which it meets the tangent to the circle drawn at the extremity of the vertical radius.

NOTE. The foregoing definitions are called the **line definitions** of the trigonometric functions. These are evidently dependent upon and follow from the ratio definitions. The ratio definitions are usually preferable but it often gives a pupil far clearer conceptions of the significance of the functions to see them represented by real lines. The two sets of definitions are in no way conflicting. Sometimes it is convenient to use one set and sometimes the other. We shall now establish them for an angle in quadrant II.



39. If we consider an ∠x in quadrant II, we make the same constructions as for the angle in quadrant I, and have the same three similar right ⊿s. The terminal line *OP* is prolonged through *O* to meet the tangent drawn from *A*, at the point *R*.



$$\sin x = \frac{TP}{1} = TP.$$

$$\cos x = \frac{OT}{1} = OT.$$

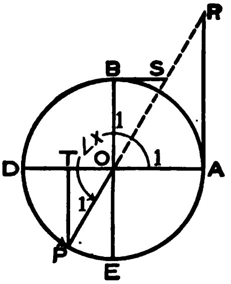
$$\tan x = \frac{TP}{OT} = \frac{AR}{OA} = \frac{AR}{1} = AR$$

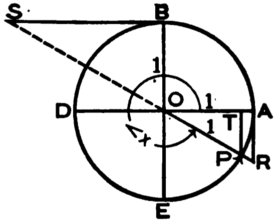
(sim. ▲, Geom. 323, 3).

$$\cot x = \frac{OT}{PT} = \frac{BS}{OB} = \frac{BS}{1} = BS.$$

$$\sec x = \frac{OP}{OT} = \frac{OR}{OA} = \frac{OR}{1} = OR. \qquad \csc x = \frac{OP}{PT} = \frac{OS}{OB} = \frac{OS}{1} = OS.$$

EXERCISE 19





If the $\angle x$ is in quadrant III, prove:

1. $\sin x = PT.$
2. $\cos x = OT.$
3. $\tan x = AR.$
4. $\cot x = BS.$
5. $\sec x = OR.$
6. $\csc x = OS.$

7 If the angle is in quadrant IV, prove the same six relations.

40. The **signs** of the **functions** of angles in the several quadrants are readily determined by the direction from the axes of reference and from the center. The sines and tangents are drawn perpendicular to the initial diameter, and they are positive, if above, negative, if below it.

The cosines and cotangents are drawn perpendicular to the vertical diameter, and they are positive, if on the right, negative, if on the left.

The secants and cosecants are drawn from the center along the terminal line (or the terminal line produced through the center), and they are positive if on the terminal line itself, negative if in the opposite direction.

The sine and cosecant of any angle agree in sign. (§ 12.)

The cosine and secant of any angle agree in sign. (§ 12.)

The tangent and cotangent of any angle agree in sign.

EXERCISE 20

1. **Draw a figure showing all the functions of an angle in the second quadrant. Write the names of each on the proper line (like the second figure of § 38) and underscore those functions that are negative.**

2. Do the same for an angle in quadrant III.

3. Do the same for an angle in quadrant IV.

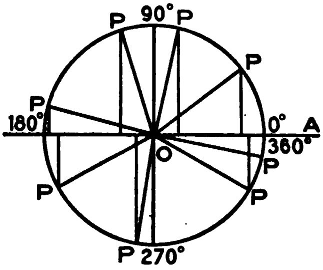
Tell the signs of the following functions:

4. sin 100°. +
5. cos 150°. -
6. tan 240° +
7. sec 210°. -
8. cot 300°. -
9. csc 280°. -
10. cot 135°. -
11. csc 120°. +
12. sin 200°. -
13. cos 250° -
14. tan 290°. -
15. sec 330°. +
16. tan 95°. -
17. sec 165° -
18. cot 190°. +
19. csc 225° -
20. sin 350° -
21. cos 310°. +
22. sin (– 30°). -
23. tan (– 100°). +
24. cos (– 210°). -
25. sin ($180° + x$). -
26. cot ($270° - \theta$). +
27. tan ($180° - A$) -

28. Verify the following table:

	QUADRANT I	QUADRANT II	QUADRANT III	QUADRANT IV
Sine and Cosecant	+	+	–	–
Cosine and Secant	+	–	–	+
Tangent and Cotangent	+	–	+	–

41. Changes in the values of the functions. Let us consider the $\angle AOP$ ($= \angle x$) to be generated by the terminal line OP, and varying from 0° to 360°. From either set of definitions it is evident that:



I. The $\sin 0° = 0$; $\sin x$ is positive and increasing as $\angle x$ increases from 0° to 90°; $\sin 90° = 1$. Sin x is positive and decreasing as $\angle x$ increases from 90° to 180°; $\sin 180° = 0$. Sin x is negative and increasing * (in absolute value) as x increases from 180° to 270°; $\sin 270° = -1$. Sin x is negative and decreasing * (in absolute value) as $\angle x$ increases from 270° to 360°; $\sin 360° = 0$.

II. The $\cos 0° = 1$; $\cos x$ is positive and decreasing as $\angle x$ increases from 0° to 90°; $\cos 90° = 0$. Cos x is negative and increasing (in absolute value) * as $\angle x$ increases from 90° to 180°; $\cos 180° = -1$. Cos x is negative and decreasing (in absolute value) * as $\angle x$ increases from 180° to 270°; $\cos 270° = 0$. Cos x is positive and increasing as $\angle x$ increases from 270° to 360°; $\cos 360° = 1$.

III. The $\tan 0° = 0$; $\tan x$ is positive and increasing as $\angle x$ increases from 0° to 90°; $\tan 90° = \infty$. Tan x is negative and decreasing (in absolute value) * as $\angle x$ increases from 90° to 180°; $\tan 180° = 0$. Tan x is positive and increasing as $\angle x$ increases from 180° to 270°; $\tan 270° = \infty$. Tan x is negative and decreasing (in absolute value) * as $\angle x$ increases from 270° to 360°; $\tan 360° = 0$.

The pupil should familiarize himself with the behavior of the individual functions of a variable angle in a complete revolution. He should, as an exercise, write out a similar paragraph for the cotangent, for the secant, and for the cosecant.

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* In numerical value, regardless of the sign.

Note. The student should review the notes of §§ 18 and 19, and understand that as the variable angle increases in the first quadrant toward 90° or decreases toward 90° in the second quadrant, the tangent becomes greater than any number that can be previously assigned. That is the meaning of the statement, $\tan 90° = \infty$. In the same way the following statements are to be understood: $\tan 270° = \infty$; $\cot 0° = \infty$; $\cot 180° = \infty$; $\sec 90° = \infty$; $\sec 270° = \infty$; $\csc 0° = \infty$; $\csc 180° = \infty$.

The results of these investigations are tabulated on p. 58.

42. In general, there are two angles less than 360° corresponding to each possible numerical value of a function.

Example. If $\sin x = \frac{2}{3}$, the $\angle x$ is either in the first or second quadrant. But if we know $\sin x = \frac{2}{3}$ and $\tan x$ is negative, the $\angle x$ must be in the second quadrant, and cannot be in the first.

EXERCISE 21

Tell the quadrant in which each of the following angles lies:

1. $\sin x = -\frac{1}{2}$ and $\cos x$ is negative. 3
2. $\tan x = \frac{3}{4}$ and $\sin x$ is negative. 3
3. $\cos x = \frac{2}{3}$ and $\tan x$ is negative. 4
4. $\tan x = 1$ and $\sec x$ is negative. 3
5. $\sec x = -5$ and $\sin x$ is positive. 2
6. $\sec \theta = 6$ and $\csc \theta$ is negative. 4
7. $\csc \beta$ is positive and $\cos \beta$ is negative. 2
8. $\cot \phi$ is negative and $\csc \phi$ is positive. 2
9. $\sin x = -\frac{1}{3}$. 34
10. $\tan x = \frac{2}{5}$. 1 3
11. $\cot x = -3$. 2 4
12. $\cos A = -\frac{1}{4}$. 2 3
13. $\csc y = 10$. 1 2
14. $\sin \theta = \frac{3}{5}$. 1 2
15. $\sec \theta = 20$. 1 4
16. $\tan M = -\frac{3}{4}$. 2 4

Find the value of each:

17. $\sin \overset{1}{90°} + \cos \overset{-1}{180°} - \sin \overset{-1}{270°}$. +1
18. $\cos \overset{1}{0°} + 2 \csc \overset{1}{90°} + 3 \cos \overset{-1}{180°}$. 0
19. $\tan \overset{0}{180°} - 3 \tan 0° + 2 \sec \overset{-1}{180°}$. −2
20. $5 \csc \overset{1}{90°} + 5 \sin 180° + 2 \sin \overset{-1}{270°}$.
21. $3 \cos \overset{-1}{180°} + 4 \sin \overset{1}{90°} + \cot 270° - \sec \overset{-1}{180°}$.
22. $\sin^2 30° + \cos^2 30° + \tan^2 30° + \sec^2 30°$. $\frac{5}{3}$

	0°	QUADRANT I		90°	QUADRANT II		180°	QUADRANT III		270°	QUADRANT IV		360°
		Sign	Behavior *		Sign	Behavior *		Sign	Behavior *		Sign	Behavior *	
Sine	0	+	Increasing	1	+	Decreasing	0	–	Increasing	– 1	–	Decreasing	0
Cosine	1	+	Decreasing	0	–	Increasing	– 1	–	Decreasing	0	+	Increasing	1
Tangent	0	+	Increasing	∞	–	Decreasing	0	+	Increasing	∞	–	Decreasing	0
Cotangent	∞	+	Decreasing	0	–	Increasing	∞	+	Decreasing	0	–	Increasing	∞
Secant	1	+	Increasing	∞	–	Decreasing	– 1	–	Increasing	∞	+	Decreasing	1
Cosecant	∞	+	Decreasing	1	+	Increasing	∞	–	Decreasing	– 1	–	Increasing	∞

* This refers to its variation independent of sign, and answers the question : How is its numerical value changing ?

23. $\overset{\frac{1}{2}\sqrt{3}}{\sin^2 60°} + \overset{\frac{1}{2}}{\cos^2 60°} - \overset{\sqrt{3}}{\tan^2 60°} + \overset{2}{\sec^2 60°}$. 2

24. $\overset{\frac{1}{2}}{\sin 30°}\ \overset{-1}{\cos 180°} + \overset{0}{\cos 90°}\ \overset{-1}{\sin 270°} - \overset{0}{\cos 180°}\ \overset{\infty}{\tan 180°}$. $-\frac{1}{2}$

25. $a\overset{0}{\cos 90°} + b\overset{0}{\sin 180°} + c\overset{0}{\tan 0°}$. 0

26. $a\overset{-1}{\sin 270°} + b\overset{-1}{\cos 180°} - c\overset{0}{\tan 180°} - d\overset{0}{\cot 270°}$. $-(a+b)$

Are the following equations possible?

27. $\sin A = 2$.

28. $\cos x = -\frac{3}{2}$.

29. $\tan x = \frac{5}{7}$. ✓

30. $\sec x = \frac{1}{3}$.

31. $\cot x = 5$. ✓

32. $\csc x = -\frac{1}{4}$.

33. $\sin x = \sqrt{2}$.

34. $\cos x = -\sqrt{3}$.

35. $\sec x = \frac{1}{2}\sqrt{2}$.

36. $\csc x = \frac{1}{2}\sqrt{3}$.

37. $\tan x = -10$. ✓

38. $\cot x = -\frac{1}{10}$. ✓

39. Between what extreme limits are all values of the sines of angles? Of the cosines of angles? 1 & −1, [illegible]

40. How are the values of the secants of angles limited? Of the cosecants of angles?

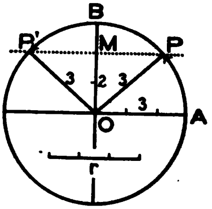
41. Are there limitations to the values that the tangent of an angle may have? The cotangent? [illegible]

42. Discuss the changes in the value of vers x as x increases from 0° to 360°. Discuss, similarly, the covers x. 1 1, 2
0, 1, 2, 1

43. Construction of angles. It is possible to construct the angles corresponding to the known value of any function.

Ex. 1. Construct $\angle x$, if $\sin x = \frac{2}{3}$.

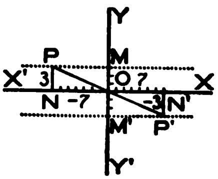
Construction: Here the ordinate = 2 and the distance = 3. $\angle x$ may be in either the first or second quadrant. Using a radius = 3 units, describe a ⊙. Draw the two ⊥ diameters as usual. On the radius OB take $OM = 2$ units and draw, through M, $PP' \parallel$ to AO. Now draw OP and OP'. The $\angle AOP$ and $\angle AOP'$ are the required ∠s.



Proof: $\sin AOP = \frac{2}{3}$. $\therefore \angle x$ may $= \angle AOP$.

$\sin AOP' = \frac{2}{3}$. $\therefore \angle x$ may $= \angle AOP$. Compare § 10.

Ex. **2.** Construct $\angle A$ if $\tan A = -\frac{3}{7}$.



The distance is not known, hence we will not draw a ⊙, but only the ⊥ axes.

The $\angle A$ is in quadrant II or IV.

$$\tan A = \frac{3}{-7} \text{ or } \frac{-3}{7}.$$

On OY take $OM = 3$ units and on OY' take $OM' = 3$ units.

Draw through M and M' lines ∥ to XX'. On OX' take $ON = 7$ units, and on OX take $ON' = 7$ units. Erect at N and N' lines ⊥ to XX' meeting the ∥s at P and P' respectively.

Draw PO and OP'. The ∡s XOP and XOP' are the required angles.

Proof. $\tan\angle XOP = -\frac{3}{7}$. $\therefore \angle A$ may $= \angle XOP$.

$\tan\angle XOP' = -\frac{3}{7}$. $\therefore \angle A$ may $= \angle XOP'$.

EXERCISE 22

Construct the angles corresponding to the following functions:

1. $\sin x = \frac{1}{3}$.

2. $\cos x = -\frac{3}{5}$.

3. $\tan x = \frac{5}{3}$.

4. $\cos y = \frac{5}{6}$.

5. $\sin x = -\frac{1}{5}$.

6. $\tan x = -\frac{2}{3}$.

7. $\sec \theta = -3$.

8. $\cot \theta = -2$.

9. $\sec \phi = -7$.

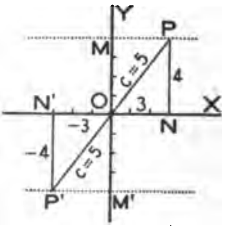
10. $\csc \alpha = 2$.

11. $\cot x = 4$.

12. $\sin \gamma = -\frac{3}{7}$.

44. If a function of any angle is given, the remaining functions may be found.

Ex. **1.** If $\tan x = \frac{3}{4}$, find the other functions of x, an angle in quadrant III.

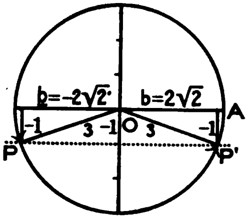


We construct the point P whose abscissa is 3 and ordinate is 4, and the point P' whose abscissa is -3, and ordinate is -4. Draw OP and OP'. The $\angle XOP'$ is the $\angle x$ desired, and the distance $OP = c = \sqrt{4^2 + 3^2} = 5$.

$$\therefore \sin x = -\tfrac{4}{5}; \quad \cos x = -\tfrac{3}{5}; \quad \cot x = \tfrac{3}{4};$$
$$\sec x = -\tfrac{5}{3}; \quad \csc x = -\tfrac{5}{4}.$$

These are the same numerical values for the functions that we should find if $\angle x$ were in the first quadrant, excepting that the signs would differ in certain cases.

Ex. **2.** If $\sin x = -\frac{1}{3}$, find all the remaining functions of $\angle x$.



The $\angle x$ may be in either quadrant III or IV. We construct $\angle x$ in both quadrants III and IV such that $\sin x = -\frac{1}{3}$. The abscissa $= b = +2\sqrt{2}$ or $-2\sqrt{2}$.

Hence, for quadrant III,

$$\sin x = -\tfrac{1}{3}; \qquad \cot x = +2\sqrt{2};$$

$$\cos x = -\tfrac{2}{3}\sqrt{2}; \qquad \sec x = -\frac{3}{2\sqrt{2}} = -\tfrac{3}{4}\sqrt{2};$$

$$\tan x = +\frac{1}{2\sqrt{2}} = +\tfrac{1}{4}\sqrt{2}; \qquad \csc x = -3.$$

And for quadrant IV,

$$\sin x = -\tfrac{1}{3}; \qquad \cot x = -2\sqrt{2};$$

$$\cos x = +\tfrac{2}{3}\sqrt{2}; \qquad \sec x = +\tfrac{3}{4}\sqrt{2};$$

$$\tan x = -\tfrac{1}{4}\sqrt{2}; \qquad \csc x = -3.$$

These results are identical with those we should find if $\angle x$ were in the first quadrant, excepting that the signs would differ in certain cases. Hence, the results may be most readily obtained by finding the numerical values as if the given angle were in the first quadrant, and prefixing the correct signs.

EXERCISE 23

Find the remaining functions of the angle in each example.

1. $\cos x = \frac{3}{5}$ and $\angle x$ is in IV.

2. $\sin x = -\frac{2}{3}$ and $\angle x$ is in III.

3. $\tan x = -1$ and $\angle x$ is in II.

4. $\sec x = 5$ and $\angle x$ is in IV.

5. $\cot x = 3$ and $\csc x$ is negative.

6. $\sin x = -\frac{1}{2}\sqrt{3}$ and $\angle x = 240°$.

7. $\csc x = 2$ and $\angle x = 150°$.

8. $\tan x = -\sqrt{3}$.

9. $\cos x = -\frac{4}{5}$.

10. $\tan x = -\frac{1}{7}$.

11. $\sin x = \frac{1}{5}$.

12. $\cot x = \sqrt{3}$.

13. $\sin x = -\frac{1}{2}$.

14. $\cos x = -\frac{1}{2}\sqrt{3}$.

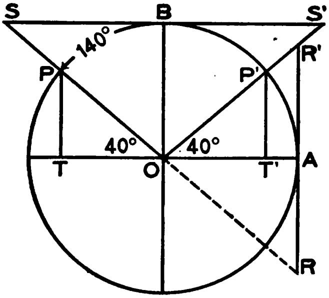
15. $\sec x = \sqrt{2}$.

16. $\csc x = -7$.

REDUCTION TO AN ACUTE ANGLE

45. Any function of an angle, however large, can be reduced to the same function, or the co-named function, of an acute angle, due regard being paid to the sign of the original function. That is, any function of an angle larger than a right angle is equal (in absolute value) to some function of an angle less than a right angle. The following reductions will illustrate and sufficiently prove the truth of this statement. The $\angle x$ is acute.

I. Let us consider an angle in the second quadrant, $180° - x$, where $x = 40°$. Draw a unit ⊙ and take $\angle AOP = 140° = 180° - 40°$. Draw $\angle AOP' = 40°$ and the several lines representing the functions.



$$\sin 140° = \sin (180° - 40°) = PT = P'T' = \sin 40°.$$

$$\cos 140° = \cos (180° - 40°) = OT = -OT' = -\cos 40°.$$

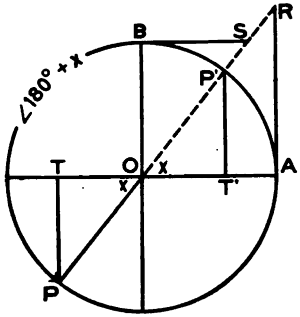
$$\tan 140° = \tan (180° - 40°) = AR = -AR' = -\tan 40°.$$

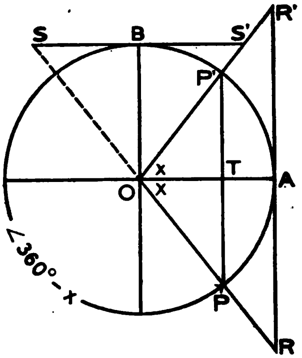
$$\cot 140° = \cot (180° - 40°) = BS = -BS' = -\cot 40°.$$

$$\sec 140° = \sec (180° - 40°) = OR = -OR' = -\sec 40°.$$

$$\csc 140° = \csc (180° - 40°) = OS = OS' = \csc 40°.$$

II. Let us consider the functions of $\angle AOP$, an angle in the third quadrant, and therefore $= 180° + x$.





$$\sin(180° + x) = PT = -P'T' = -\sin x.$$
$$\cos(180° + x) = OT = -OT' = -\cos x.$$

$$\tan(180° + x) = AR = \tan x.$$
$$\cot(180° + x) = BS = \cot x.$$

$$\sec(180° + x) = OR = -\sec x.$$
$$\csc(180° + x) = OS = -\csc x.$$

III. Let us consider the functions of $\angle AOP$, an angle in the fourth quadrant, and therefore $= 360° - x$.

$$\sin(360° - x) = PT = -P'T = -\sin x.$$
$$\cos(360° - x) = OT = \cos x.$$

$$\tan(360° - x) = AR = -AR' = -\tan x.$$
$$\cot(360° - x) = BS = -BS' = -\cot x.$$

$$\sec(360° - x) = OR = OR' = \sec x.$$
$$\csc(360° - x) = OS = -OS' = -\csc x.$$

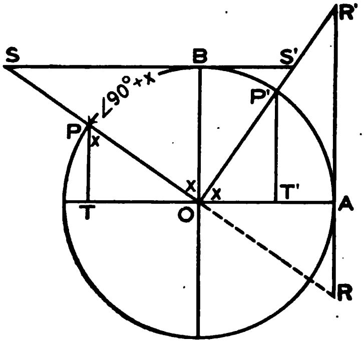
From these three illustrations, including all angles of quadrants II, III, and IV, provided ∠ x is acute, it is obvious that:

1. **Any function of $(180° - x) = \pm$ the same function of x;**
2. **Any function of $(180° + x) = \pm$ the same function of x;**
3. **Any function of $(360° - x) = \pm$ the same function of x.**

46. We shall now investigate the functions of $(90° \pm x)$ and of $(270° \pm x)$. In § 13 we discovered that any function of $(90° - x) =$ the co-named function of x.

Let ∠ AOP in quadrant II $= (90° + x)$.

$\triangle OPT = \triangle OP'T'$, $\triangle OAR = \triangle OBS'$, $\triangle OBS = \triangle OAR'$ (Geom. 72, 74).



$$\sin (90° + x) = PT = OT' = \cos x.$$

$$\cos (90° + x) = OT = -P'T' = -\sin x.$$

$$\tan (90° + x) = AR = -BS' = -\cot x.$$

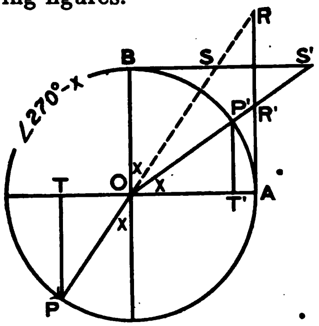
$$\cot (90° + x) = BS = -AR' = -\tan x.$$

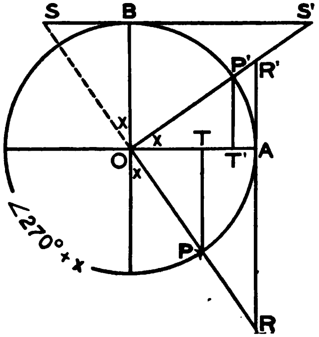
$$\sec (90° + x) = OR = -OS' = -\csc x.$$

$$\csc (90° + x) = OS = OR' = \sec x.$$

EXERCISE 24

1. Namė the three pairs of equal right $\triangle$ in each of the following figures.





Prove the following equations true:

2. $\sin(270° - x) = -\cos x$.
3. $\cos(270° - x) = -\sin x$.
4. $\tan(270° - x) = \cot x$.
5. $\cot(270° - x) = \tan x$.
6. $\sec(270° - x) = -\csc x$.
7. $\csc(270° - x) = -\sec x$.
8. $\sin(270° + x) = -\cos x$.
9. $\cos(270° + x) = \sin x$.
10. $\tan(270° + x) = -\cot x$.
11. $\cot(270° + x) = -\tan x$.
12. $\sec(270° + x) = \csc x$.
13. $\csc(270° + x) = -\sec x$.

From these illustrations including all angles of quadrants II, III, IV, provided $\angle x$ is acute, it is obvious that:

4. **Any function of $(90° \pm x) = \pm$ the co-function of x.**

5. **Any function of $(270° \pm x) = \pm$ the co-function of x.**

47. Functions of a negative angle. A negative angle has the same terminal line as the corresponding positive angle, and therefore the values of the functions of a negative angle are the same as the values of the functions of the corresponding positive angle. That is,

6. **Any function of $(-x)$** = same function of $(360° - x)$
= **$\pm$ the same function of x.**

EXAMPLES

Ex. 1. What angle has the same sine as 80°?

$\sin 80°$ is positive, and hence the required $\angle$ is in quadrant II
$\sin(180° - 80°) = \sin 80°$.
$\therefore \sin 100° = \sin 80°$. $\therefore$ the required $\angle = 100°$.

Ex. 2. What angle has the same cosine as 150°?

$\cos 150° = \cos(180° - 30°) = -\cos 30°$. This is negative.
$\therefore$ the required $\angle$ is in quadrant III.
$\cos(180° + 30°) = -\cos 30°$.
$\therefore \cos 210° = \cos 150°$. $\therefore$ the required $\angle = 210°$.

Ex. 3. Reduce tan 155°.

$\tan 155° = \tan(180° - 25°) = -\tan 25°$.

Ex. 4. Reduce cos (– 122°).

$\cos(-122°) = \cos(360° - 122°) = \cos 238° = \cos(270° - 32°)$
$= -\sin 32°$.

Ex. 5. Reduce sin (990° – *A*).

$\sin(990° - A) = \sin(270° - A) = -\cos A$.

Ex. 6. Reduce sin 120°.

$\sin 120° = \sin(90° + 30°) = \cos 30° = \frac{1}{2}\sqrt{3}$.

Ex. 7. Reduce tan 300°.

$\tan 300° = \tan(360° - 60°) = -\tan 60° = -\sqrt{3}$.

Ex. 8. Reduce sec (– 210°).

$\sec(-210°) = \sec(360° - 210°) = \sec 150° = \sec(180° - 30°)$
$= -\sec 30° = -\frac{2}{3}\sqrt{3}$.

Ex. 9. Reduce tan (θ – 270°).

$\tan(-270° + \theta) = \tan(360° - 270° + \theta) = \tan(90° + \theta) = -\cot\theta$

Ex. 10. If sin 156° = .41, find tan 156°.

$\sin 156° = \sin 24° = .41$.
$\therefore$ by right $\triangle$ (§ 11), $\tan 24° = .44^{+}$. $\tan 156° = -\tan 24° = -.44^{+}$.

Ex. 11. If tan 228° = 1.11, find cos 132°.

$\tan 228° = \tan 48° = 1.11$.
$\therefore$ by right $\triangle$, $\cos 48° = .67^{+}$. $\cos 132° = -\cos 48° = -.67^{+}$.

EXERCISE 25

Find the angle (less than 360°) that has the same

1. sine as 150°.
2. cosine as 50°.
3. tangent as 80°.
4. cotangent as 130°
5. secant as 200°.
6. cosecant as 200°.
7. cosine as 120°.
8. tangent as 300°.
9. cotangent as 240°
10. sine as 260°.
11. cosecant as 135°.
12. secant as 330°.

Reduce to simplest form; that is, to a function of an angle less than 45°:

13. $\sin 173^\circ$
14. $\tan 100^\circ$
15. $\cos 195^\circ$.
16. $\sec 257^\circ$.
17. $\cot 228^\circ$.
18. $\sec 206^\circ$.
19. $\tan 295^\circ$.
20. $\cos 310^\circ$.
21. $\sin 350^\circ$.
22. $\cot 345^\circ$.
23. $\sin 426^\circ$.
24. $\cos 507^\circ$.
25. $\tan 531^\circ$.
26. $\sin(-160^\circ)$.
27. $\cos(-230^\circ)$.
28. $\tan(-15^\circ)$.
29. $\sec(-311^\circ)$.
30. $\csc(-46^\circ)$.
31. $\cot(-165^\circ)$.
32. $\sin 125^\circ 10'$.
33. $\cos 156^\circ 8'$.
34. $\tan 250^\circ 30'$.
35. $\cot 205^\circ 20'$.
36. $\sec 293^\circ 40'$
37. $\csc 320^\circ 50'$.
38. $\sin(-40^\circ 50')$.
39. $\cos(-140^\circ 20')$.

Reduce to simplest form:

40. $\sin(540^\circ + A)$.
41. $\cos(450^\circ + \theta)$.
42. $\tan(-90^\circ - x)$.
43. $\sin(\theta - 90^\circ)$.
44. $\cos(-180^\circ - \phi)$.
45. $\tan(360^\circ + \theta)$.
46. $\sin(-270^\circ - \theta)$
47. $\tan(x - 180^\circ)$.
48. $\cos(x - 630^\circ)$.

Find the numerical value of each:

49. $\sin 150^\circ$.
50. $\cos 120^\circ$.
51. $\tan 120^\circ$.
52. $\sin 135^\circ$.
53. $\cos 135^\circ$.
54. $\tan 135^\circ$.
55. $\sin 225^\circ$.
56. $\cos 210^\circ$.

57. $\tan 210°$.
58. $\csc 210°$.
59. $\tan 240°$.
60. $\sin 240°$.
61. $\sec 225°$.
62. $\cos 300°$.
63. $\cot 330°$.
64. $\sin 330°$.
65. $\tan 420°$.
66. $\sec 495°$.
67. $\sin(-45°)$.
68. $\cos(-30°)$.
69. $\tan(-60°)$.
70. $\cot(-120°)$.
71. $\sec(-120°)$.
72. $\tan(-210°)$.

Simplify the following expressions:

73. $\sin(90° + A) + \cos(180° - A)$.

74. $\sin(180° + A) - \cos(270° - A)$.

75. $\sin(270° + A) + \cos(360° - A)$.

76. $\tan(270° - A) - \cot(180° - A)$.

77. $\sin(90° + A)\cos(360° - A) + \sin(180° + A)\cos(90° + A)$.

78. $\sin(180° + A)\sin(180° - A) - \sin(270° + A)\sin(270° - A)$.

79. $\cos(180° - A)\cos(180° + A) - \cos(270° - A)\cos(270° + A)$.

80. $\sec(180° - A)\sec(180° + A) + \tan(360° + A)\tan(360° - A)$.

81. $\sec(90° + A)\sec(90° - A) - \tan(270° + A)\tan(270° - A)$.

82. $\sin 120° + \cos 150° + \tan 135° - \cos 180°$.

83. $\cos 120° - \sin 150° - \tan 135° - \sec 240°$.

84. $\sin 225° - \tan 210° + \cos 315° + \sec 150°$.

85. $\cos 240° + \sin^2 240° - \tan^2 240° + \sec^2 225°$.

86. $\cos 270° - \tan 315° - \csc^2 300° + \cos 300° + \tan^2 330°$.

87. $\sin 135° \cos 225° - \tan 150° \cot 240° + \sec 120° \csc 240°$.

88. $\sin 150° \cos 315° - \tan 135° \cot 210° - \sec 240 \csc 300°$.

89. If $\cos 144° = -.81$, find $\sin 36°$; $\tan 36°$; $\sin 216°$; $\cos 216°$.

90. If $\sin 160° = .34$, find $\cos 200°$; $\cot 200°$; $\tan(-20°)$.

91. If $\tan 130° = -1.19$, find $\sec 310°$; $\cos 230°$; $\tan(-50°)$.

92. If $\cos 322 = .79$, find $\sin 38°$; $\sin 52°$; $\cos 142°$; $\tan 128°$

93. Find the values of the six functions of $(-30°)$.

94. Find the values of the six functions of $(-60°)$.

95. Find the values of the six functions of (-45°).

96. Find the values of the six functions of $(-\theta)$.

97. If $\angle A$ is in quadrant II, in which quadrant is $\frac{1}{2}A$? 1 $2A$? 4

98. If $\angle x$ is in quadrant III, in which quadrant is $\frac{1}{2}x$? 2 $2x$? 6

99. If $\angle \alpha$ is in quadrant IV, in which quadrant is $\frac{1}{2}\alpha$? 2 2α? 3

100. If $\angle\theta$ is in quadrant III, in which quadrant is $\frac{1}{3}\theta$? 1 $\frac{1}{2}(90^\circ+\theta)$?

EQUATIONS

48. We have already solved a few simple trigonometric equations that involved an unknown angle in the first quadrant (see § 21). We are now prepared to find other angles in the solution, beyond the first quadrant.

Ex. 1. Find x if $\sin x = \frac{1}{2}$.

Evidently $x = 30^\circ$ or another angle in II. $\therefore x$ may $= 150^\circ$. Each of these may be increased by 360°. $\therefore x$ may $= 390^\circ, 510^\circ$, etc.

Ex. 2. Solve: $4\cos^2\theta = 1$ for all values of $\theta < 360^\circ$.

$\cos\theta = \pm\frac{1}{2}$. For $\cos\theta = +\frac{1}{2}$, θ may $= 60^\circ$ or 300°.
For $\cos\theta = -\frac{1}{2}$, θ may $= 120^\circ$ or 240°.

Ex. 3. Solve: $\sin 3x = \frac{1}{2}\sqrt{3}$ for all values of $x < 360^\circ$.

$$3x = 60^\circ, 120^\circ, \text{ or } 420^\circ, 480^\circ, \text{ or } 780^\circ, 840^\circ.$$
$$\therefore x = 20^\circ, 40^\circ, 140^\circ, 160^\circ, 260^\circ, 280^\circ.$$

Ex. 4. Solve: $2\sin^2 x + \sqrt{2}\cos x = 2$.

$2(1-\cos^2 x) + \sqrt{2}\cos x = 2$. $\therefore 2\cos^2 x - \sqrt{2}\cos x = 0$.
Hence, $\cos x = 0$ or $\cos x = \frac{1}{2}\sqrt{2}$.
$\therefore x = 90^\circ, 270^\circ$, or $x = 45^\circ, 315^\circ$.

Ex. 5. Find $\angle A$ if $\sin 200^\circ = \sin A$.

$\sin 200^\circ = -\sin 20^\circ$. $\therefore$ the required $\angle$ is in IV.
$\sin(360^\circ - 20^\circ) = -\sin 20^\circ$. $\therefore \sin 200^\circ = \sin 340^\circ$. $\therefore A = 340^\circ$.

Ex. 6. Find the values of $\angle\theta$ if $\cos\theta = -\cos 305^\circ$.

$-\cos 305^\circ = -\cos 55^\circ$. $\therefore$ θ is in II or III.

For II, $\cos(180^\circ - 55^\circ) = -\cos 55^\circ$. $\therefore \cos 125^\circ = -\cos 305^\circ$. $\therefore \theta = 125^\circ$.

For III, $\cos(180^\circ + 55^\circ) = -\cos 55^\circ$. $\therefore \cos 235^\circ = -\cos 305^\circ$. $\therefore \theta = 235^\circ$

Ex. 7. Find the values of x if $\sin x = -\cos 140^\circ$.

$-\cos 140^\circ = -\cos(90^\circ + 50^\circ) = \sin 50^\circ$. $\therefore$ $\angle x$ is in I or II.

Evidently $\sin(180^\circ - 50^\circ) = \sin 50^\circ$ and $x = 50^\circ$ or 130°.

EXERCISE 26

Solve the following equations, obtaining all possible values less than 360° for the unknown angle in each:

1. $\cos x = \pm\frac{1}{2}\sqrt{2}$.
2. $\sin x = \pm 1$.
3. $\tan x = \pm 1$.
4. $\cot x = \pm\sqrt{3}$.
5. $\sec x = \pm 2$.
6. $4\cos^2 x = 3$.
7. $4\sin^2 x = 3$.
8. $3\csc^2 x = 4$.
9. $\sin x = 0$.
10. $\tan^2 x = 3$.
11. $4\sin^2 x = 1$.
12. $\sin^2 x = \sin x$.
13. $3\sin^2 x = \cos^2 x$.
14. $\tan x = 3\cot x$.
15. $\sqrt{2}\sin\theta = \tan\theta$.
16. $3\sin\theta = 2\cos^2\theta$.
17. $\tan x + 2\sin x = 0$.
18. $\sin 2x = \pm\frac{1}{2}$.
19. $\cos 3x = \pm\frac{1}{2}$.
20. $\tan 3x = \pm\frac{1}{3}\sqrt{3}$.
21. $\sin 5x = \pm 1$.
22. $\cos 4\theta = \pm\frac{1}{2}\sqrt{2}$.
23. $\sin\frac{1}{2}A = \pm\frac{1}{2}\sqrt{3}$.
24. $\cos\frac{1}{2}\beta = \pm\frac{1}{2}\sqrt{2}$.
25. $\tan^2\frac{1}{3}\theta = 1$.
26. $\cot^2\frac{2}{3}\theta = 3$.
27. $\sin 2x\cos 3x = 0$.
28. $\cot x + \sqrt{2}\cos x = 0$.
29. $\sin x\sec x = -\sqrt{3}$.
30. $\tan\phi\sec\phi = -2\sqrt{3}$.
31. $4\sin y\cos y = -\sqrt{3}$.

(solving a radical equation it is well [illegible] not be penalty [illegible] unless they check

32. $2(\sin\theta + \csc\theta) = -5.$

33. $3(\sec^2 x + \cot^2 x) = 13.$

34. $4 \sin^2 x \cos x = 3 \cos x.$

35. $2 \cos^2 x = 1 + \cos x.$

36. $2 \sin^2 x = 1 - \sin x.$

37. $2 \sin x \cos x = -\cos x.$

38. $2 \cos^2\theta + \sqrt{2} \sin\theta = 2.$

39. $3 \tan^2 A - 4 \sin^2 A = 1.$

40. $\cos^2\theta - \sqrt{3} \cos\theta + \frac{3}{4} = 0$

41. $2 \sin\theta + \cos\theta \cot\theta = \frac{5}{2}.$

42. $\tan^2 x - \sec x = 1.$

43. $2 \sin^2 x + \cos^2 x = \frac{3}{2}.$

44. $\tan x + \cot x = 2.$

45. $4(\sin^2 x - \cos x) = 1.$

46. $2 \sin x + \cot x = 1 + 2 \cos x.$

47. $\sin x = \sin 30°.$

48. $\cos\theta = -\cos 45°.$

49. $\tan A = -\tan 60°.$

50. $\cot\alpha = \tan 30°.$

51. $\cos x = -\sin 60°.$

52. $\sin x = \sin 40°.$

53. $\sin x = \sin 117°.$

54. $\cos x = \cos 117°.$

55. $\tan x = \tan 117°.$

56. $\sin\theta = \sin 220°.$

57. $\cos\theta = \cos 215°.$

58. $\sec y = \sec 100°.$

59. $\sin x = -\sin 145°.$

60. $\cos x = -\cos 145°.$

61. $\tan x = -\tan 145°.$

62. $\sin\theta = -\sin 230°.$

63. $\cos\alpha = -\cos 230°.$

64. $\cot\lambda = -\cot 255°.$

65. $\sin\gamma = -\sin 302°.$

66. $\cos\phi = -\cos 302°.$

67. $\sin x = \cos 220°.$

68. $\sin\theta = \cos 305°.$

69. $\cos x = \sin 128°.$

70. $\cos A = -\sin 280°$

71. $\tan\theta = \cot 105°.$

72. $\tan x = -\cot 350°.$

73. $\sec x = \csc 205°.$

74. $\sec x = -\csc 333°.$

ELEMENTARY RELATIONS OF THE FUNCTIONS OF ANY ANGLE

49. In §§ 12 and 15 we established several relations among the several functions of an acute angle; that is, an angle in the first quadrant. To establish these relations for all angles, we need only prove them true if $\angle x$ is an angle (*a*) in II, (*b*) in III, and (*c*) in IV.

Given: $\angle x = 180° - A$, and $\angle A$ an acute $\angle$.

To Prove: (1) $\sin x \csc x = 1, \cos x \sec x = 1, \tan x \cot x = 1.$

(2) $\sin^2 x + \cos^2 x = 1$; (3) $\tan^2 x + 1 = \sec^2 x$;

$$(4)\ \tan x = \frac{\sin x}{\cos x}, \text{ etc.}$$

Proof: Let us substitute $180° - A$ instead of x and simplify the resulting expression.

(1) $\sin x \csc x = \sin(180° - A)\csc(180° - A) = \sin A \csc A = 1.$ Q.E.D.

(2) $\sin^2 x + \cos^2 x = \sin^2(180° - A) + \cos^2(180° - A)$
$= \sin^2 A + \cos^2 A = 1.$ Q.E.D.

(3) $\tan^2 x + 1 = \tan^2(180° - A) + 1 = (-\tan A)^2 + 1$
$= \tan^2 A + 1,$

and $\sec^2 x = \sec^2(180° - A) = (-\sec A)^2 = \sec^2 A.$

But $\tan^2 A + 1 = \sec^2 A$ because $\angle A$ is acute.

$\therefore \tan^2 x + 1 = \sec^2 x$ (Ax. 1). Q.E.D.

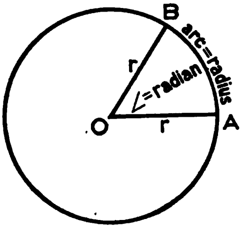
Similarly, the others are proved when $x = 180° - A$, or $180° + A$ or $360° - A$.

The actual proofs are left as an exercise.

CIRCULAR MEASURE OF ANGLES

50. There are two distinct and common methods employed in the measurement of angles. One of these is called the **sexagesimal method**, in which the units are the familiar **degree**, **minute**, and **second**. This is the method used in practice and in most elementary mathematics. The other of these is called the **circular method**, in which the unit is the **radian**. This is the method used in theoretical treatment and in nearly all of the higher mathematics.

The **radian** is an angle at the center of a circle, that intercepts an arc equal in length to the radius.



If arc AB = a radius, $\angle AOB$ = a radian.

51. Problem. To find the number of degrees in one radian and the number of radians in one degree.

The circumference of ⊙ $O = 2\pi r$ (Geom. 444).

The circumference of a ⊙ = 360 degrees of arc.

$\therefore 2\pi r = 360$ degrees of arc (Ax. 1).

$\therefore$ the arc $r = \dfrac{180}{\pi}$ degrees of arc. But a central angle contains the same number of degrees as its intercepted arc (Geom. 245).

$$\therefore \angle AOB, \text{ or 1 radian} = \frac{180°}{\pi}.$$

Clearing this equation, π radians = 180°.

or $$1° = \frac{\pi}{180} \text{ radian.}$$

52. Theorem. The radian is a constant; that is, its value is the same for all circles.

Proof: The quantity π is a constant.

$\therefore \dfrac{180°}{\pi}$ is a constant. Therefore the radian is a constant.

The numerical value of 1 radian $= \dfrac{180°}{3.14159265} = 57.29578$ degrees.

The numerical value of 1 degree $= \dfrac{3.14159265}{180} = .01745$ radian.

53. Theorem. The circular measure of an angle is the ratio of the length of the intercepted arc to the length of the radius.

Given: Central $\angle x$ intercepting arc a on the circumference of a $\odot$ whose radius is r.

To Prove: $\angle x = \frac{a}{r}$ radians.

Proof: In the same $\odot$ the radian intercepts an arc $= r$.

$\therefore \frac{\angle x}{1 \text{ radian}} = \frac{a}{r}$ (Geom. 244). $\quad \therefore \angle x = \frac{a}{r}$ radians.

Note. This is merely an application of the definition of measuring a quantity. (See Geom. 237.)

54. By virtue of § 51 we derive these rules:

1. **To reduce a certain number of radians to degrees multiply by $\frac{180}{\pi}$.**

2. **To reduce a certain number of degrees to radians multiply by $\frac{\pi}{180}$.**

Note. π is a constant and $= 3.14159265+$.
π radians means 3.14159265 radians and $= 180°$.
$\pi - 1$ radians means 2.14159265 radians and $= 122.7°$.
π radians is the same angle as 180°.
$\frac{1}{2}\pi$ radians is the same angle as 90°.
2π radians is the same angle as 360°.
But π is not 180°. If the word "radians" is understood after the symbol "π," *then*, π is equal to 180°.

Ex. 1. Change $\frac{3}{2}\pi$ radians and 1.5 radians to degrees.

$\frac{3}{2}\pi \times \frac{180}{\pi} = 270°. \qquad \frac{3}{2} \times \frac{180}{\pi} = \frac{270°}{\pi} = 270° \div 3.1416 = 85°.9.$

Ex. 2. Change 60° and 315° to radians.

$60° \times \frac{\pi}{180} = \frac{\pi}{3}\text{r}. \qquad 315° \times \frac{\pi}{180} = \frac{7\pi}{4}\text{ r}.$

Ex. 3. How many radians are there in the angle at the center of a circle whose radius is 15 in. if the length of the arc is 6 in.? Express this angle in degrees also.

The $\angle = \frac{6}{15} = \frac{2}{5}$ r. $\qquad \frac{2}{5} \times \frac{180}{\pi} = \frac{72}{3.1416} = 22^\circ.9.$

Ex. 4. If the minute hand of a town clock is 4 ft. long, how far does its outer end move in 20 min.?

The central $\angle$ generated $= 120^\circ = \frac{2\pi}{3}$ r.

$\therefore \frac{2\pi}{3} = \frac{\text{arc}}{\text{radius}}$ (§ 53). $\qquad \therefore \frac{2\pi}{3} = \frac{\text{arc}}{4}. \qquad \therefore \text{arc} = \frac{8\pi}{3}$ ft.

That is, arc $= \frac{8}{3} \times 3.1416 = 8.3776$ ft.

EXERCISE 27

1. Express each of the following angles in degrees:

$\frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{5}, \frac{\pi}{6}, \frac{\pi}{8}, \frac{5\pi}{9}, \frac{\pi}{10}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{5\pi}{6}, \frac{3\pi}{8}, \frac{11\pi}{18},$

$\pi, 2\pi, 3\pi, 4\pi, \pi+1, \pi-1, \frac{\pi-1}{2}, \frac{2\pi+1}{3}, 2.5\pi, \frac{1}{3}, 2, 1.4,$ 3, 3.5, .5, .1, 1.1 radians.

2. Express each of the following angles in radians:

40°, 60°, 90°, 150°, 160°, 300°, 270°, 306°, 63°, 210°, 140°, − 120°, − 330°, − 200°, 405°, 540°, 630°.

Find the actual numerical value for each of the following:

3. $\sin\frac{\pi}{6}$; $\cos\frac{\pi}{6}$; $\tan\frac{\pi}{4}$; $\sec\frac{\pi}{3}$; $\tan\frac{\pi}{3}$; $\sin\frac{\pi}{2}$; $\cos\frac{\pi}{2}$.

4. $\cos\frac{2\pi}{3}$; $\csc\frac{3\pi}{4}$; $\cot\frac{5\pi}{6}$; $\cos\frac{5\pi}{6}$; $\sin\frac{7\pi}{6}$; $\cos\frac{4\pi}{3}$.

5. $\sin\frac{5\pi}{4}$; $\tan\frac{3\pi}{2}$; $\sin\frac{3\pi}{2}$; $\cot\frac{5\pi}{3}$; $\sec\frac{5\pi}{3}$; $\sin\frac{11\pi}{6}$; $\cos\frac{11\pi}{6}$.

Reduce:

6. $\sin\left(\frac{\pi}{2}+x\right)$; $\cos\left(\frac{\pi}{2}-x\right)$; $\tan(\pi+\theta)$; $\cot(\pi-\theta)$.

7. $\sec\left(\frac{3\pi}{2}-\phi\right)$; $\csc\left(\frac{3\pi}{2}+\phi\right)$; $\cot(2\pi-a)$; $\cos(\pi+x)$.

8. What is the number of radians in the central angle intercepting these arcs:

(*a*) arc 10 in., radius 3 in.?	(*c*) arc 6 in., radius 3 ft.?
(*b*) arc 3 in., radius 6 in.?	(*d*) arc .45 ft., radius 6 yd.?

9. Find the corresponding radii if an arc of 12 ft. is intercepted by a central angle of 1 r.; of 2 r.; of 4 r.; of $\frac{1}{2}$ r.; of $\frac{2}{3}$ r.; of $3\frac{2}{3}$ r.

10. In a circle whose radius is 6 in., find the lengths of the arcs intercepted by the following central ∡s: 2 r.; $1\frac{1}{2}$ r.; $\frac{2}{3}$ r.; 3 r.; π r.; $\frac{1}{2}(\pi-1)$ r.; $2\frac{1}{3}$ r.; $\frac{3}{8}$ r.

11. An arc 33 yd. long is intercepted by a central angle containing $3\frac{2}{3}$ radians. What is the length of the radius?

12. Express in radians: 22° 30′; 67° 30′; 78° 45′; 101° 15′; 151° 52′ 30″.

13. A central angle is equal to $2\frac{1}{3}$ radians. What is the radius if the intercepted arc is 9 ft.? If the arc is $1\frac{1}{6}$ mi.? If the arc is 10.5 m.? If the arc is 3.57 yd.?

GRAPHS OF THE TRIGONOMETRIC FUNCTIONS

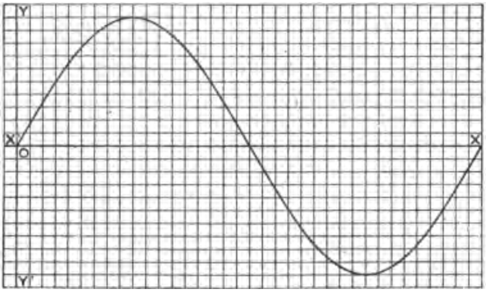
55. The equation $\sin x = y$ contains two variables, and for every value of the angle x there is a corresponding value of y. It is possible, therefore, to represent all the values of any trigonometric function by a curve, and the character of this curve will reflect the behavior of the function as the angle varies.

Ex. 1. Let it be required to draw the curve of sines.

Suppose several equal distances taken on the axes of X and Y from their intersection, each of those on the X-axis representing 10°, and each one on the Y-axis representing .1. From the table of natural sines, one finds:

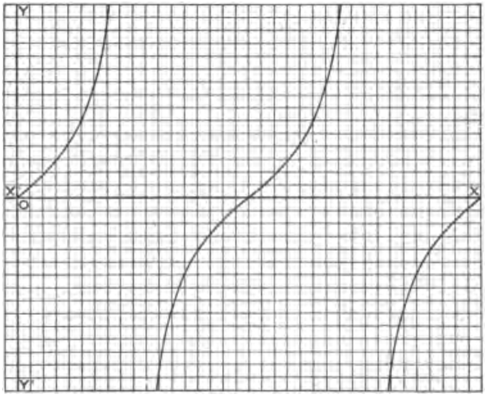
$\sin 0° = 0$
$\sin 10° = .17 +$
$\sin 20° = .34 +$
$\sin 30° = .50$
$\sin 40° = .64 +$
$\sin 50° = .76 +$
$\sin 60° = .86 +$
$\sin 70° = .94 -$
$\sin 80° = .98 +$
$\sin 90° = 1.00$
$\sin 100° = .98 +$
$\sin 110° = .94 -$
$\sin 120° = .86 +$
$\sin 130° = .76 +$
etc. etc.

By continuing this valuation of the $\sin x$ as x increases from 0° to 360° we may ascertain the ordinates of any number of points, whose abscissas are represented by the numbers of degrees in the angles. Through these several points a smooth, regular curve is drawn.



Ex. 2. Let it be required to draw the curve of tangents.

Let each unit on the X-axis represent 10°, and each unit on the Y-axis represent .2. The several values of the tangent of several angles are obtained from the table of natural tangents as before, and the corresponding points are plotted, and the curve drawn.



In curves of sine and cosine the values of the functions are always relatively small, and it is well to employ a unit of ordinates = .1 as we have. But in the other curves the values of the functions are not always small, and for them, as in the curve of tangents, a more satisfactory value for the unit of ordinates is .2.

EXERCISE 28

1. Draw the graph of cos x.
2. Draw the graph of cot x.
3. Draw the graph of sec x.
4. Draw the graph of csc x.
5. Draw the graph of $\sin x + \cos x$.
6. Draw the graph of $\sin x - \cos x$.
7. Compare the discussions of § 41 with the curves of the six functions already drawn.
8. Trace the behavior of vers x, as the angle x increases from 0° to 360°. Also, of covers x.
9. A line parallel to the X-axis intersects each curve representing a function, either twice or not at all. Interpret the significance of this, trigonometrically. Discuss the cases when this line becomes tangent to the curve.
10. A line parallel to the Y-axis intersects each curve only once. Interpret the significance of this.
11. Draw the graphs of vers x and covers x.

NOTE. The curves of the six functions could have been plotted without the use of the tables, that is geometrically, by proceeding as follows:

1. Draw a unit circle and central angles (the aliquot parts of 360°), and the function line for each angle. These are the ordinates of the various points to be plotted, and the unit of ordinates is thus the radius, a unit of length.

2. Take a distance upon the X-axis $= 2\pi$ units (approximately), the length of the circumference of the unit circle. Divide this distance into 36 equal parts, and let each represent $\frac{\pi}{18}$ radian. Thus the unit of abscissas is the radian.

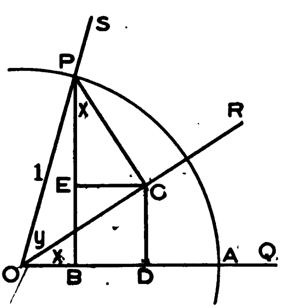
The advantage of this method is its uniform scale.

The disadvantage is the labor attached to the necessary constructions.

CHAPTER IV

GENERAL FORMULAS

56. Problem. To derive formulas for $\sin(x+y)$, $\cos(x+y)$, $\tan(x+y)$, and $\cot(x+y)$.



Given: $\angle QOR = x$, $\angle ROS = y$, $\angle QOS = x + y$.

Required: Formulas for $\sin(x+y)$, $\cos(x+y)$, $\tan(x+y)$, and $\cot(x+y)$.

Derivation: Using O as a center, describe a unit circle, cutting OQ at A, OS at P. From P draw $PB \perp$ to OQ and $PC \perp$ to OR. From C draw $CD \perp$ to OQ and $CE \perp$ to PB.

△s OCD and CPE are similar (Geom. 321). $\therefore \angle CPE = \angle x$.

$$\text{In } \triangle OCD, \begin{cases} \sin x = \dfrac{CD}{OC}. & \therefore CD = \sin x \cdot OC. \\ \cos x = \dfrac{OD}{OC}. & \therefore OD = \cos x \cdot OC. \end{cases}$$

$$\text{In } \triangle CPE, \begin{cases} \sin x = \dfrac{CE}{PC}. & \therefore CE = \sin x \cdot PC. \\ \cos x = \dfrac{PE}{PC}. & \therefore PE = \cos x \cdot PC. \end{cases}$$

$$\text{In } \triangle OPC, \begin{cases} \sin y = \dfrac{PC}{1} = PC. \\ \cos y = \dfrac{OC}{1} = OC. \end{cases}$$

Now, $\sin(x+y) = PB = CD + PE = \sin x \cdot OC + \cos x \cdot PC.$

$$\therefore \sin(x+y) = \sin x \cos y + \cos x \sin y. \qquad . \quad . \quad (1)$$

Also, $\cos(x+y) = OB = OD - CE = \cos x \cdot OC - \sin x \cdot PC.$

$$\therefore \cos(x+y) = \cos x \cos y - \sin x \sin y. \qquad . \quad . \quad (2)$$

$$\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}.$$

Divide each term of this fraction by $\cos x \cos y$.

$$\tan(x+y) = \frac{\dfrac{\sin x \cos y}{\cos x \cos y} + \dfrac{\cos x \sin y}{\cos x \cos y}}{\dfrac{\cos x \cos y}{\cos x \cos y} - \dfrac{\sin x \sin y}{\cos x \cos y}} = \frac{\dfrac{\sin x}{\cos x} + \dfrac{\sin y}{\cos y}}{1 - \dfrac{\sin x}{\cos x} \cdot \dfrac{\sin y}{\cos y}}.$$

$$\therefore \tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \cdot \tan y}. \qquad . \quad . \quad . \quad . \quad (3)$$

$$\cot(x+y) = \frac{\cos(x+y)}{\sin(x+y)} = \frac{\cos x \cos y - \sin x \sin y}{\sin x \cos y + \cos x \sin y}.$$

Divide each member of this fraction by $\sin x \sin y$.

$$\cot(x+y) = \frac{\dfrac{\cos x \cos y}{\sin x \sin y} - \dfrac{\sin x \sin y}{\sin x \sin y}}{\dfrac{\sin x \cos y}{\sin x \sin y} + \dfrac{\cos x \sin y}{\sin x \sin y}} = \frac{\dfrac{\cos x}{\sin x} \cdot \dfrac{\cos y}{\sin y} - 1}{\dfrac{\cos y}{\sin y} + \dfrac{\cos x}{\sin x}}.$$

$$\therefore \cot(x+y) = \frac{\cot x \cot y - 1}{\cot y + \cot x}. \qquad . \quad . \quad . \quad . \quad (4)$$

57. Formulas (1) and (2) have been proved for the case that x, y, and $x+y$ are all acute angles. The formulas are true relations for all values of the angles involved. To establish their validity, in general, we may proceed as follows, always assuming that $\angle A < 90^\circ$ and $\angle B < 90^\circ$.

I. Suppose $x < 90^\circ$, $y < 90^\circ$, $x + y > 90^\circ$, and that

$$\left.\begin{array}{rlcrl} x = & 90^\circ - A & & \therefore A = & 90^\circ - x \\ y = & 90^\circ - B & & \therefore B = & 90^\circ - y \\ \hline x + y = & 180^\circ - \overline{A + B} & \text{and} & A + B = & 180^\circ - \overline{x + y} \end{array}\right\} \begin{array}{l} \text{But } x + y > 90^\circ. \\ \therefore A + B < 90^\circ. \end{array}$$

$$\sin(x+y) = \sin(180^\circ - \overline{A+B}) = \sin(A+B)$$
$$= \sin A \cos B + \cos A \sin B \quad (A,\ B,\ A+B \text{ are each } < 90^\circ.)$$
$$= \sin(90^\circ - x)\cos(90^\circ - y) + \cos(90^\circ - x)\sin(90^\circ - y)$$
(Substitution.)
$$= \cos x \sin y + \sin x \cos y. \quad \text{(See § 13.)}$$

$$\cos(x+y) = \cos(180^\circ - \overline{A+B}) = -\cos(A+B)$$
$$= -\cos A \cos B + \sin A \sin B \quad (A,\ B,\ A+B \text{ are each } < 90.^\circ)$$
$$= -\cos(90^\circ - x)\cos(90^\circ - y) + \sin(90^\circ - x)\sin(90^\circ - y)$$
(Substitution.)
$$= -\sin x \sin y + \cos x \cos y. \quad \text{(See § 13.)}$$

II. Suppose $x = 90^\circ + A$, $y < 90^\circ$. $\therefore x + y > 90^\circ$.

$\therefore x + y = 90^\circ + \overline{A + y}$; $A = x - 90^\circ$; $y = y$.

$$\sin(x+y) = \sin(90^\circ + \overline{A+y}) = \cos(A+y) \quad \text{(See § 46.)}$$
$$= \cos A \cos y - \sin A \sin y$$
$$= \cos(x - 90^\circ)\cos y - \sin(x - 90^\circ)\sin y$$
$$= \sin x \cos y + \cos x \sin y.$$

$$\cos(x+y) = \cos(90^\circ + \overline{A+y}) = -\sin(A+y) = \text{etc.}$$

III. By supposing $x = 90^\circ + A$, $y = 90^\circ + B$, $x + y > 180^\circ$, and so on, the process of proving these formulas true for all quadrants in which x, y, and $x + y$ may be, is precisely as above. If A is $< 90^\circ$, $90^\circ + A$, $180^\circ + A$, $270^\circ + A$ include all angles in quadrants II, III, and IV., etc.

EXERCISE 29

Prove formulas (1) and (2),

1. If x is in quadrant III and y in quadrant II.
2. If x is in quadrant III and y in quadrant IV.
3. If x is in quadrant I and y in quadrant III.

58. Applications. Ex. 1. Find cos 75°.

$$\cos 75^\circ = \cos(45^\circ + 30^\circ) = \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ$$
$$= \tfrac{1}{2}\sqrt{2} \cdot \tfrac{1}{2}\sqrt{3} - \tfrac{1}{2}\sqrt{2} \cdot \tfrac{1}{2} = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \tfrac{1}{4}(\sqrt{6} - \sqrt{2}).$$

Ex. 2. Develop $\sin(270^\circ + x)$.

$$\sin(270^\circ + x) = \sin 270^\circ \cos x + \cos 270^\circ \sin x$$
$$= -1 \cdot \cos x + 0 \cdot \sin x = -\cos x. \quad \text{(Compare § 46.)}$$

Ex. 3. If $\sin A = \frac{2}{3}$ and $\sin B = \frac{1}{6}\sqrt{6}$, find the value of $\tan(A + B)$.

First, $\tan A = \dfrac{2}{\sqrt{5}} = \dfrac{2}{5}\sqrt{5}$ and $\tan B = \dfrac{1}{5}\sqrt{5}$.

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\frac{2}{5}\sqrt{5} + \frac{1}{5}\sqrt{5}}{1 - \frac{2}{5}\sqrt{5} \cdot \frac{1}{5}\sqrt{5}} = \sqrt{5}.$$

Ex. 4. Develop $\sin(x - A)$; that is, $\sin[x + (-A)]$.

$$\sin(x - A) = \sin[x + (-A)] = \sin x \cos(-A) + \cos x \sin(-A)$$
$$= \sin x \cos A + \cos x(-\sin A)$$
$$= \sin x \cos A - \cos x \sin A.$$

45+30

60+45

EXERCISE 30

1. Find $\sin 75^\circ$.
2. Find $\tan 75^\circ$.
3. Find $\cot 75^\circ$.
4. Find $\sin 105^\circ$.
5. Find $\cos 105^\circ$.
6. Find $\tan 105^\circ$.

Develop:

7. $\sin(45^\circ + x)$.
8. $\tan(45^\circ + x)$.
9. $\cos(30^\circ + A)$.
10. $\cos\left(\frac{\pi}{2} + y\right)$.
11. $\sin(A + A)$.
12. $\cos(\theta + \theta)$.
13. $\sin(150^\circ + x)$.
14. $\tan\left(\frac{4\pi}{3} + \phi\right)$.

If $\sin x = \frac{3}{5}$, and x is acute, find the value of:

15. $\sin(90^\circ + x)$.
16. $\cos(60^\circ + x)$.
17. $\cos(x + x)$.
18. $\tan(180^\circ + x)$.

If $\sin A = \frac{4}{5}$ and $\tan B = \frac{5}{12}$ and A and B are each acute, find:

19. $\sin(A + B)$.
20. $\cos(A + B)$.
21. $\tan(A + B)$.
22. $\cot(A + B)$.

If $\sin\theta=\frac{1}{5}\sqrt{5}$, $\tan\phi=\frac{1}{3}$, θ is in II, ϕ is in III, find:

23. $\sin(\theta+\phi)$. **24.** $\cos(\theta+\phi)$. **25.** $\tan(\theta+\phi)$.

Prove:

26. $\sin(60^\circ+x)-\cos(30^\circ+x)=\sin x$.

27. $\cos(60^\circ+A)+\sin(30^\circ+A)=\cos A$.

28. $1+\tan(45^\circ+M)=\dfrac{2}{1-\tan M}$.

29. $\sin(x+y)\cos x-\cos(x+y)\sin x=\sin y$.

30. $\cos(m+n)\cos m+\sin(m+n)\sin m=\cos n$.

31. $\sin(x+y+z)=\sin x\cos y\cos z+\cos x\sin y\cos z+\cos x\cos y\sin z-\sin x\sin y\sin z$.

32. $\cos(x+y+z)=\cos x\cos y\cos z-\sin x\sin y\cos z-\sin x\cos y\sin z-\cos x\sin y\sin z$.

33. $\cos 15^\circ+\sin 75^\circ-\sin 105^\circ+\cos 165^\circ=0$.

34. $\tan A+\tan B=\dfrac{\sin(A+B)}{\cos A\cos B}$.

59. Problem. To derive formulas for $\sin 2x$, $\cos 2x$, $\tan 2x$, and $\cot 2x$.

Derivation: $2x=x+x$; hence if $y=x$ in the formulas of § 56, they will become the desired formulas.

$$\sin(x+x)=\sin x\cos x+\cos x\sin x.$$

$$\therefore \sin 2x=2\sin x\cos x. \quad \ldots\ldots \quad (5)$$

$$\cos(x+x)=\cos x\cos x-\sin x\sin x.$$

$$\therefore \cos 2x=\cos^2 x-\sin^2 x \quad \ldots\ldots \quad (6)$$

$$\text{Or } \cos 2x=(1-\sin^2 x)-\sin^2 x.$$

$$\therefore \cos 2x=1-2\sin^2 x \quad \ldots\ldots \quad (7)$$

$$\text{Or } \cos 2x=\cos^2 x-(1-\cos^2 x).$$

$$\therefore \cos 2x=2\cos^2 x-1 \quad \ldots\ldots \quad (8)$$

$$\tan(x+x)=\frac{\tan x+\tan x}{1-\tan x\tan x}.$$

$$\therefore \tan 2x=\frac{2\tan x}{1-\tan^2 x} \quad \ldots\ldots \quad (9)$$

$$\cot(x + x) = \frac{\cot x \cot x - 1}{\cot x + \cot x}.$$

$$\therefore \cot 2x = \frac{\cot^2 x - 1}{2 \cot x} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (10)$$

60. Applications. Ex. 1. Develop tan 120°; or, find tan 2 (60°).

$$\tan 120° = \tan 2\,(60°) = \frac{2 \tan 60°}{1 - \tan^2 60°} = \frac{2\sqrt{3}}{1-3} = -\sqrt{3}.$$

Ex. 2. If $\sin x = \frac{5}{13}$, find $\cos 2x$ and $\tan 2x$.

$$\cos 2x = 1 - 2\sin^2 x = \frac{119}{169}. \quad \tan 2x = \frac{2 \tan x}{1 - \tan^2 x} = \frac{120}{119}.$$

Ex. 3. Develop $\sin 3x$, that is, $\sin(2x + x)$, in terms of $\sin x$.

$$\begin{aligned}\sin 3x &= \sin(2x + x) = \sin 2x \cos x + \cos 2x \sin x\\ &= 2 \sin x \cos x \cdot \cos x + (1 - 2\sin^2 x)\sin x\\ &= 2 \sin x \cos^2 x + \sin x - 2 \sin^3 x\\ &= 2 \sin x(1 - \sin^2 x) + \sin x - 2\sin^3 x\\ &= 2 \sin x - 2\sin^3 x + \sin x - 2 \sin^3 x = \mathbf{3 \sin x - 4 \sin^3 x}.\end{aligned}$$

EXERCISE 31

Find, by use of the formulas of § 59:

1. sin 60°. **4.** cot 60°. **6.** cos 120°.
2. cos 60°. **5.** sin 120°. **7.** cot 120°.
3. tan 60°. **8.** cos 100° in terms of sin 50°
9. cos 100° in terms of cos 50°.
10. tan 80° in terms of tan 40°

If $\sin x = \frac{3}{5}$, 4 find the numerical value of:

11. $\sin 2x$. **12.** $\cos 2x$. **13.** $\tan 2x$. **14.** $\cot 2x$.

If $\cos \theta = -\frac{3}{5}$ and θ is in quadrant II, find:

15. $\sin 2\theta$. **16.** $\cos 2\theta$. **17.** $\tan 2\theta$. **18.** $\cot 2\theta$.

If $\tan\alpha = \frac{1}{2}$ and α is in quadrant III, find:

19. $\sin 2\alpha$. **20.** $\cos 2\alpha$. **21.** $\tan 2\alpha$. **22.** $\cot 2\alpha$.

Prove: **23.** $\sin 3x = 3\sin x - 4\sin^3 x$ by developing $\sin(x+2x)$.

24. $\cos 3x = 4\cos^3 x - 3\cos x$.

25. $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$.

26. $\sin 4A = 4\sin A\cos A - 8\sin^3 A\cos A$.

27. $\sin 4x = 8\cos^3 x\sin x - 4\cos x\sin x$.

28. $\cos 4\alpha = 1 - 8\sin^2\alpha + 8\sin^4\alpha$.

29. $\cos 4\theta = 1 - 8\cos^2\theta + 8\cos^4\theta$.

30. $\tan 4A = \dfrac{4\tan A - 4\tan^3 A}{1 - 6\tan^2 A + \tan^4 A}$.

31. $\dfrac{2\tan x}{1+\tan^2 x} = \sin 2x$.

32. $\dfrac{1+\tan^2 x}{1-\tan^2 x} = \sec 2x$.

33. $\dfrac{\sec^2 A}{2-\sec^2 A} = \sec 2A$.

34. $\tan B + \cot B = 2\csc 2B$.

35. $\cot x - \tan x = 2\cot 2x$.

36. $\dfrac{\sin 2\theta}{\sin\theta} - \dfrac{\cos 2\theta}{\cos\theta} = \sec\theta$.

37. $\dfrac{\sin 2y}{1+\cos 2y} = \tan y$.

38. $\dfrac{1-\cos 2x}{\sin 2x} = \tan x$.

39. $\cos^4\theta - \sin^4\theta = \cos 2\theta$.

40. $(\sin A + \cos A)^2 = 1 + \sin 2A$.

41. $\dfrac{\sin x + \sin 2x}{1 + \cos x + \cos 2x} = \tan x$.

42. $\dfrac{\sin 3x}{\sin x} - \dfrac{\cos 3x}{\cos x} = 2$.

61. PROBLEM. To derive formulas for $\sin\frac{1}{2}\theta$, $\cos\frac{1}{2}\theta$, $\tan\frac{1}{2}\theta$, and $\cot\frac{1}{2}\theta$.

Derivation: From (7), $\cos 2x = 1 - 2\sin^2 x$.

From (8), $\cos 2x = 2\cos^2 x - 1$.

$$\therefore \sin^2 x = \frac{1-\cos 2x}{2}.$$

$$\therefore \sin x = \pm\sqrt{\frac{1-\cos 2x}{2}}.$$

Also, $$\cos^2 x = \frac{1+\cos 2x}{2}.$$

$$\therefore \cos x = \pm\sqrt{\frac{1+\cos 2x}{2}}.$$

Now, let $x = \frac{1}{2}\theta$. $\therefore 2x = \theta$. Substituting,

$$\sin\tfrac{1}{2}\theta = \pm\sqrt{\frac{1-\cos\theta}{2}} \quad \dots\dots\dots \quad (11)$$

$$\cos\tfrac{1}{2}\theta = \pm\sqrt{\frac{1+\cos\theta}{2}} \quad \dots\dots\dots \quad (12)$$

$$\tan\tfrac{1}{2}\theta = \frac{\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \quad \dots \quad (13)$$

$$\cot\tfrac{1}{2}\theta = \frac{\cos\frac{1}{2}\theta}{\sin\frac{1}{2}\theta} = \pm\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} \quad \dots \quad (14)$$

The ambiguous sign is dependent upon the quadrant in which $\frac{1}{2}\theta$ is.

62. Applications. Ex. 1. Develop tan 30°; that is find tan $\frac{1}{2}(60°)$.

$$\tan 30° = \tan\frac{1}{2}(60°) = +\sqrt{\frac{1-\cos 60°}{1+\cos 60°}} = +\sqrt{\frac{1-\frac{1}{2}}{1+\frac{1}{2}}} = +\sqrt{\frac{1}{3}} = +\frac{1}{3}\sqrt{3}.$$

Ex. 2. Reduce the formula for $\tan\frac{1}{2}\theta$ to one having a rational denominator.

$$\tan\tfrac{1}{2}\theta = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \times \sqrt{\frac{1-\cos\theta}{1-\cos\theta}} = \sqrt{\frac{(1-\cos\theta)^2}{\sin^2\theta}} = \frac{1-\cos\theta}{\sin\theta}. = \frac{\sin\theta}{1+\cos\theta}$$

Ex. 3. Prove $(\tan\frac{1}{2}x + \cot\frac{1}{2}x)^2 = 4\csc^2 x$.

$$(\tan\tfrac{1}{2}x + \cot\tfrac{1}{2}x)^2 = \left(\sqrt{\frac{1-\cos x}{1+\cos x}} + \sqrt{\frac{1+\cos x}{1-\cos x}}\right)^2$$

$$= \frac{1-\cos x}{1+\cos x} + 2 + \frac{1+\cos x}{1-\cos x}$$

$$= \frac{4}{\sin^2 x} = 4\csc^2 x.$$

EXERCISE 32

Find, by use of the formulas of § 62:

1. sin 30°. **3.** sin 15°. **5.** tan 15°.

2. cos 45°. **4.** cos 15°. **6.** cot 45°.

7. Find all the functions of 22° 30′.

8. Express the functions of 5° in terms of cos 10°.

9. Express the functions of 1° in terms of cos 2°.

10. Express the functions of α in terms of cos 2α.

If $\sin x = \frac{7}{25}$ and x is in quadrant I, find:

11. $\sin \frac{1}{2} x$. **12.** $\cos \frac{1}{2} x$. **13.** $\tan \frac{1}{2} x$. **14.** $\cot \frac{1}{2} x$.

If $\tan \alpha = -\frac{5}{12}$ and α is in quadrant II, find:

15. $\sin \frac{1}{2} \alpha$. **16.** $\cos \frac{1}{2} \alpha$. **17.** $\tan \frac{1}{2} \alpha$. **18.** $\sec \frac{1}{2} \alpha$.

If $\cot \theta = 2\sqrt{2}$ and θ is in quadrant III, find:

19. $\sin \frac{1}{2} \theta$. **20.** $\cos \frac{1}{2} \theta$. **21.** $\tan \frac{1}{2} \theta$. ~~**22.**~~ $\csc \frac{1}{2} \theta$.

In which quadrants may $\angle A$ be, if:

5678 **23.** $\sin \frac{1}{2} A$ is negative? **25.** $\tan \frac{1}{2} A$ is positive? 1 2

3415 **24.** $\cos \frac{1}{2} A$ is negative? **26.** $\tan \frac{1}{2} A$ is negative? 234

Prove:

27. $2 \cos^2 \frac{1}{2} A = 1 + \cos A$. **28.** $2 \sin^2 \frac{1}{2} x = 1 - \cos x$.

29. $\tan \frac{1}{2} \theta = \frac{\sin \theta}{1 + \cos \theta}$; or $= \frac{1 - \cos \theta}{\sin \theta}$; or $= \csc \theta - \cot \theta$.

30. $\cot \frac{1}{2} \theta = \frac{\sin \theta}{1 - \cos \theta}$; or $= \frac{1 + \cos \theta}{\sin \theta}$; or $= \csc \theta + \cot \theta$.

31. In a right triangle $\cos A = \frac{b}{c}$ and $\cos B = \frac{a}{c}$. Whence, show that $\sin \frac{1}{2} A = \sqrt{\frac{c - b}{2c}}$ (see § 61). Also $\sin \frac{1}{2} B = \sqrt{\frac{c - a}{2c}}$.

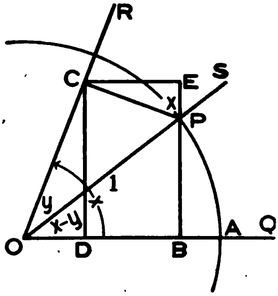
NOTE. This formula is useful in solving right triangles provided an angle is near 90°. Interpolation for angles of this size by the ordinary logarithmic tables is inaccurate.

Find one acute angle in each of the following right triangles:

32. $a = 1.7$; $c = 2016.5$. **33.** $b = .35$; $c = 3762.8$.

34. $a = .048$; $c = 863.6$.

63. Problem. To derive formulas for sin $(x - y)$, cos $(x - y)$, tan $(x - y)$, and cot $(x - y)$.



Given: $\angle QOR = x$; $\angle ROS = y$; $\angle QOS = x - y$.

Required: Formulas for $\sin(x-y)$, $\cos(x-y)$, $\tan(x-y)$, $\cot(x-y)$.

Derivation: Using O as a center, describe a unit circle cutting OQ at A, OS at P. From P draw $PB \perp$ to OQ and $PC \perp$ to OR. From C draw $CD \perp$ to OQ and $CE \perp$ to PB.

▲ OCD and CPE are similar (Geom. 321). $\therefore \angle CPE = \angle x$.

$$\text{In } \triangle OCD \begin{cases} \sin x = \dfrac{CD}{OC}. & \therefore CD = \sin x \cdot OC. \\ \cos x = \dfrac{OD}{OC}. & \therefore OD = \cos x \cdot OC. \end{cases}$$

$$\text{In } \triangle CPE \begin{cases} \sin x = \dfrac{CE}{PC}. & \therefore CE = \sin x \cdot PC. \\ \cos x = \dfrac{PE}{PC}. & \therefore PE = \cos x \cdot PC. \end{cases}$$

$$\text{In } \triangle OPC \begin{cases} \sin y = \dfrac{PC}{1} = PC. \\ \cos y = \dfrac{OC}{1} = OC. \end{cases}$$

$$\sin(x-y) = PB = CD - PE = \sin x \cdot OC - \cos x \cdot PC.$$

$$\therefore \sin(x-y) = \sin x \cos y - \cos x \sin y \quad . \quad . \quad (15)$$

$$\cos(x-y) = OB = OD + CE = \cos x \cdot OC + \sin x \cdot PC.$$

$$\therefore \cos(x-y) = \cos x \cos y + \sin x \sin y \quad . \quad . \quad (16)$$

$$\tan(x-y)=\frac{\sin(x-y)}{\cos(x-y)}=\frac{\sin x\cos y-\cos x\sin y}{\cos x\cos y+\sin x\sin y}.$$

Divide each term of this fraction by $\cos x\cos y$ and reduce:

$$\therefore \tan(x-y)=\frac{\tan x-\tan y}{1+\tan x\tan y} \quad . \quad . \quad . \quad . \quad (17)$$

Likewise,
$$\cot(x-y)=\frac{\cot x\cot y+1}{\cot y-\cot x} \quad . \quad . \quad . \quad . \quad (18)$$

64. Formulas (15) and (16) have been proved for the particular case that x, y, and $x-y$ are all acute angles. The formulas may be established, in general, for all values of the angles involved, by employing formulas (1) and (2) which have been established for all values, in § 57.

Thus, $\sin(x+A)=\sin x\cos A+\cos x\sin A$,

and $\cos(x+A)=\cos x\cos A-\sin x\sin A$,

are true for all values of x, A, and $x+A$. Substitute $(-y)$ for A in them.

$$\sin[x+(-y)]=\sin x\cos(-y)+\cos x\sin(-y),$$

and
$$\cos[x+(-y)]=\cos x\cos(-y)-\sin x\sin(-y).$$

$$\therefore \sin(x-y)=\sin x\cos y-\cos x\sin y,$$

and
$$\cos(x-y)=\cos x\cos y+\sin x\sin y.$$

Or these may be proved universally, as was done in § 57.

Note

A relation among the trigonometric functions which is true for all angles is called an **identity**. All formulas involving trigonometric functions are trigonometric identities.

EXERCISE 33

Develop:

1. $\sin(30^\circ-x)$.
2. $\cos(180^\circ-x)$.
3. $\tan(45^\circ-\theta)$.
4. $\sin\left(\frac{5\pi}{6}-A\right)$.
5. $\cos\left(x-\frac{5\pi}{6}\right)$.
6. $\tan(60^\circ-45^\circ)$.
7. Find all the functions of 15°, using $(45^\circ-30^\circ)$.

If $\sin x = .6$ and $\cos y = \frac{15}{17}$, find:

8. $\sin(x-y)$.
9. $\cos(x-y)$.
10. $\tan(x-y)$.
11. $\cot(x-y)$.

If $\cos A = \frac{4}{5}$ and $\tan B = \frac{1}{7}$, find:

12. $\sin(A-B)$.
13. $\cos(A-B)$.
14. $\tan(A-B)$.
15. $\sec(A-B)$.

If $\sin x = -\frac{1}{3}$, x in quadrant III, $\tan y = \frac{1}{2}$, y in quadrant I, find:

16. $\sin(x-y)$.
17. $\cos(x-y)$.
18. $\tan(x-y)$.
19. $\csc(x-y)$.

If $\cos x = -\frac{5}{13}$, x in quadrant II, find:

20. $\sin(270^\circ - x)$.
21. $\cos(x - 60^\circ)$.
22. $\tan(240^\circ - x)$.
23. $\sin 2x$.
24. $\cos \frac{1}{2}x$.
25. $\sin(2x - \frac{1}{2}x)$.

Prove:

26. $\cos(90^\circ - x) + \sin(180^\circ - x) = 2\sin x$.
27. $\sin(60^\circ - \theta) + \cos(30^\circ - \theta) = \sqrt{3}\cos\theta$.
28. $\tan(60^\circ + \phi) - \cot(30^\circ - \phi) = 0$.
29. $\sin(m+n) + \sin(m-n) = 2\sin m \cos n$.
30. $\cos(m+n) - \cos(m-n) = -2\sin m \sin n$.
31. $\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y = \cos^2 y - \cos^2 x$.
32. $\cos(x+y)\cos(x-y) = \cos^2 x - \sin^2 y = \cos^2 y - \sin^2 x$.
33. $\tan(\alpha+\beta)\tan(\alpha-\beta) = \dfrac{\tan^2\alpha - \tan^2\beta}{1 - \tan^2\alpha\tan^2\beta}$.
34. $(\cos x + \sin x)(\cos y + \sin y) = \cos(x-y) + \sin(x+y)$.
35. $2\sin(45^\circ - \alpha)\cos(45^\circ + \beta) = \cos(\alpha - \beta) - \sin(\alpha + \beta)$.
36. $2\sin(45^\circ + \alpha)\cos(45^\circ + \beta) = \cos(\alpha + \beta) + \sin(\alpha - \beta)$
37. Develop $\tan[x + (-y)]$ and thus derive formula (17).
38. Develop $\cot[x + (-y)]$ and thus derive formula (18).

65. PROBLEM. To derive formulas for sin $A \pm$ sin B and cos $A \pm$ cos B.

Derivation:

$$\begin{array}{lrcl}
\text{From (1),} & \sin(x+y) & = & \sin x\cos y + \cos x\sin y \\
\text{From (15),} & \sin(x-y) & = & \sin x\cos y - \cos x\sin y \\
\hline
\text{Adding,} & \sin(x+y)+\sin(x-y) & = & 2\sin x\cos y. \\
\text{Subtracting,} & \sin(x+y)-\sin(x-y) & = & 2\cos x\sin y.
\end{array}$$

$$\begin{array}{lll}
\text{Suppose} & x+y=A & \\
\text{and} & x-y=B & \\
\hline
 & 2x = A+B. & \therefore x=\tfrac{1}{2}(A+B). \\
 & 2y = A-B. & \therefore y=\tfrac{1}{2}(A-B).
\end{array}$$

Making these substitutions above, one obtains,

$$\sin A + \sin B = 2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B). \quad . \quad (19)$$

$$\sin A - \sin B = 2\cos\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B). \quad . \quad (20)$$

$$\begin{array}{lrcl}
\text{Again, from (2),} & \cos(x+y) & = & \cos x\cos y - \sin x\sin y \\
\text{and from (16),} & \cos(x-y) & = & \cos x\cos y + \sin x\sin y \\
\hline
\text{Adding,} & \cos(x+y)+\cos(x-y) & = & 2\cos x\cos y. \\
\text{Subtracting,} & \cos(x+y)-\cos(x-y) & = & -2\sin x\sin y.
\end{array}$$

Substituting as before,

$$\cos A + \cos B = 2\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B). \quad . \quad . \quad (21)$$

$$\cos A - \cos B = -2\sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B). \quad . \quad . \quad (22)$$

NOTE. The student should clearly understand the distinction between formulas (1) and (19); between (2) and (21); between (15) and (20); and between (16) and (22). In one case we have a single function of a single angle $(x+y)$ or $(x-y)$. In the other case we have the sum or difference of a function of two different angles. In one case the left hand side is a monomial, and the right, a binomial. In the other, the left side is a binomial, and the right, a monomial.

66. Applications. Ex. 1. Reduce $\sin 5A + \sin 3A$ to a product.

$$\sin 5A + \sin 3A = 2\sin\tfrac{1}{2}(8A)\cos\tfrac{1}{2}(2A) = 2\sin 4A\cos A.$$

Ex. 2. Prove: $\cos 85^\circ + \cos 35^\circ = \cos 25^\circ$.

$\cos 85^\circ + \cos 35^\circ = 2\cos\frac{1}{2}(120^\circ)\cos\frac{1}{2}(50^\circ) = 2\cos 60^\circ \cos 25^\circ = \cos 25^\circ.$

Ex. 3. Prove: $\sin 10^\circ - \sin 70^\circ + \sin 130^\circ = 0$.

Formula (20) enables us to combine the first two terms:

$$\begin{aligned}\overline{\sin 10^\circ - \sin 70^\circ} + \sin 130^\circ &= 2\cos\tfrac{1}{2}(80^\circ)\sin\tfrac{1}{2}(-60^\circ) + \sin 130^\circ\\ &= -\cos 40^\circ + \sin(90^\circ + 40^\circ)\\ &= -\cos 40^\circ + \cos 40^\circ = 0.\end{aligned}$$

EXERCISE 34

Reduce each to a product:

1. $\sin 4x - \sin 2x$.
2. $\sin 5A + \sin A$.
3. $\cos 3\theta + \cos\theta$.
4. $\sin 7m - \sin 5m$.
5. $\cos 2A - \cos 6A$.
6. $\cos 5\theta - \cos\theta$.
7. $\cos 5x + \cos 9x$.
8. $\sin 80^\circ + \sin 20^\circ$.
9. $\sin\frac{1}{2}x + \sin\frac{3}{2}x$.
10. $\cos\frac{3}{2}\theta + \cos\frac{1}{2}\theta$.
11. $\cos 15^\circ - \cos 17^\circ$.
12. $\sin 4x - \sin 2y$.

Prove:

13. $\sin 40^\circ + \sin 20^\circ = \cos 10^\circ$.
14. $\cos 50^\circ + \cos 70^\circ = \cos 10^\circ$.
15. $\sin 50^\circ - \sin 10^\circ = \sqrt{3}\sin 20^\circ$.
16. $\cos 20^\circ - \cos 80^\circ = \sin 50^\circ = \cos 40^\circ$.
17. $\cos A - \cos 3A = 2\sin 2A \sin A = 4\cos A - 4\cos^3 A$.
18. $\dfrac{\sin 66^\circ + \sin 6^\circ}{\cos 66^\circ + \cos 6^\circ} = \tan 36^\circ$.
19. $\dfrac{\sin 3x + \sin 5x}{\cos 3x - \cos 5x} = \cot x$.
20. $\dfrac{\sin 75^\circ - \sin 15^\circ}{\cos 75^\circ + \cos 15^\circ} = \dfrac{1}{3}\sqrt{3}$
21. $\dfrac{\sin 15^\circ + \sin 75^\circ}{\sin 15^\circ - \sin 75^\circ} = -\sqrt{3}$.

22. $\cos 20° + \cos 100° + \cos 140° = 0.$

23. $\sin 18° + \sin 42° + \cos 168° = 0.$

24. $\sin 83° + \sin 97° + 2 \sin 263° = 0.$

25. $\cos 32° - \cos 28° + \cos 88° = 0.$

26. $\sin 3x + 2 \sin 5x + \sin 7x = 4 \sin 5x \cos^2 x.$

27. $\cos 3x + 2 \cos 5x + \cos 7x = 4 \cos 5x \cos^2 x.$

Reduce each to a product:

28. $\sin (20° + A) - \sin (20° - A).$

29. $\sin (A + 2B) + \sin (3A - 2B).$

30. $\cos (63° + x) + \cos (27° - x).$

31. $\cos (50° - \theta) - \cos (50° + \theta).$

32. $\dfrac{\sin A - \sin B}{\sin A + \sin B}$

33. $\dfrac{\sin x + \sin y}{\cos x + \cos y}.$

34. $\dfrac{\sin \theta - \sin \phi}{\cos \theta - \cos \phi}.$

67. By a simple algebraic substitution, reductions otherwise often laborious and difficult are rendered comparatively easy.

Ex. 1. Prove:

$$\cos (x + y) \cos y + \sin (x + y) \sin y = \cos x.$$

Proof: Suppose $x + y = A$, then the left member becomes:

$$\begin{aligned} \cos A \cos y + \sin A \sin y &= \cos (A - y) \\ &= \cos (\overline{x + y} - y) \\ &= \cos x. \end{aligned}$$

Ex. 2. Prove: $\dfrac{\cot \frac{1}{2} x - \tan \frac{1}{2} x}{\cot \frac{1}{2} x + \tan \frac{1}{2} x} = \cos x.$

Proof: Suppose $\frac{1}{2} x = A$. $\therefore x = 2A$. Substitute in left member:

$$\frac{\cot A - \tan A}{\cot A + \tan A} = \frac{\dfrac{\cos A}{\sin A} - \dfrac{\sin A}{\cos A}}{\dfrac{\cos A}{\sin A} + \dfrac{\sin A}{\cos A}} = \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A}$$

$$= \cos 2A = \cos x.$$

Ex. 3. Write a formula for $\tan 4x$ in terms of $\tan 2x$.

Suppose $4x = 2A$. $\therefore 2x = A$.

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A} \quad \text{[Formula (9)].}$$

$$\therefore \tan 4x = \frac{2 \tan 2x}{1 - \tan^2 2x} \quad \text{(Substitution).}$$

EXERCISE 35

Prove the following relations:

1. $\sin x \cos (x - y) - \cos x \sin (x - y) = \sin y$.
2. $\sin 4x \cos 3x + \cos 4x \sin 3x = \sin 7x$.
3. $\cos (A + B) \cos (A - B) - \sin (A + B) \cos (A - B) = \cos 2A$.
4. $2 \cos (45° + \frac{1}{2} m) \cos (45° - \frac{1}{2} m) = \cos m$.
5. $\tan (45° + \frac{1}{2} \theta) + \tan (45° - \frac{1}{2} \theta) = 2 \sec \theta$.
6. $\dfrac{1 + \tan \frac{1}{2} x}{1 - \tan \frac{1}{2} x} = \dfrac{\cos x}{1 - \sin x}$.
7. $\dfrac{2 \cot \frac{1}{2} A}{\cot^2 \frac{1}{2} A - 1} = \tan A$.
8. Write $\cos 10x$ in terms of $\cos 5x$. Of $\sin 5x$.
9. Write $\cos 2a$ in terms of $\cos 4a$.
10. Write $\tan 3a$ in terms of $\cos 6a$.
11. Write all the functions of θ in terms of $\cos \frac{1}{2} \theta$.
12. Write $\sin \theta$ and $\cos \theta$ in terms of $\sin \frac{1}{2} \theta$ and $\cos \frac{1}{2} \theta$ by use of formulas (5)–(8).

EXERCISE 36

Miscellaneous Exercise

Prove the following relations:

1. $\dfrac{\sin 2A}{1 - \cos 2A} = \cot A$.
2. $\dfrac{2 - \sec^2 x}{\sec^2 x} = \cos 2x$.
3. $\dfrac{\cos (m - n)}{\sin m \cos n} = \cot m + \tan n$.
4. $\dfrac{\cos 2\alpha}{1 + \sin 2\alpha} = \dfrac{1 - \tan \alpha}{1 + \tan \alpha}$.

5. $\dfrac{1-\tan^2\theta}{1+\tan^2\theta}=\cos 2\theta.$ **6.** $\dfrac{2\sin^3 x}{1-\cos x}=2\sin x+\sin 2x.$

7. $\dfrac{\cos^3 A+\sin^3 A}{\cos A+\sin A}=1-\frac{1}{2}\sin 2A.$

8. $\dfrac{\cos^3 x-\sin^3 x}{\cos x-\sin x}=1+\frac{1}{2}\sin 2x.$

9. $\dfrac{\sin(x+y)}{\sin(x-y)}=\dfrac{\tan x+\tan y}{\tan x-\tan y}.$ **10.** $\dfrac{\sin(x-y)}{\sin(x+y)}=\dfrac{\cot y-\cot x}{\cot y+\cot x}.$

11. $\dfrac{\sin(A+B)}{\cos(A-B)}=\dfrac{\tan A+\tan B}{1+\tan A\tan B}.$

12. $\dfrac{\cos(\theta+\phi)}{\cos(\theta-\phi)}=\dfrac{\cot\phi-\tan\theta}{\cot\phi+\tan\theta}.$ **13.** $\left(\dfrac{\tan x+1}{\tan x-1}\right)^2=\dfrac{1+\sin 2x}{1-\sin 2x}.$

14. $\dfrac{\sin m}{1+\cos m}+\dfrac{1+\cos m}{\sin m}=2\csc m.$

15. $(\sec A-\tan A)^{-1}=\sec A+\tan A.$

16. $\sin x(1+\tan x)+\cos x(1+\cot x)=\sec x+\csc x.$

17. $(1+\cot\alpha+\tan\alpha)(\sin\alpha-\cos\alpha)=\sin^2\alpha\sec\alpha-\cos^2\alpha\csc\alpha.$

18. $\dfrac{\cos(x+45^\circ)}{\cos(x-45^\circ)}=\sec 2x-\tan 2x.$

19. $(\csc x-\cot x)^2=\dfrac{1-\cos x}{1+\cos x}.$

20. $\dfrac{\sin 4A}{\sin 2A}=2\cos 2A.$ **21.** $\dfrac{\cos 2\theta}{1-\sin 2\theta}=\dfrac{1+\tan\theta}{1-\tan\theta}.$

22. $\tan 2x+\sec 2x=\dfrac{\cos x+\sin x}{\cos x-\sin x}.$

23. $\cos 5x\cos 3x+\sin 5x\sin 3x=\cos 2x.$

24. $\sin 52^\circ 30'+\sin 7^\circ 30'=\cos 22^\circ 30'.$

25. $\cos 3x-\cos 7x=2\sin 5x\sin 2x.$

26. $\sin(x-y)\cos y+\cos(x-y)\sin y=\sin x.$

27. $\sqrt{\dfrac{1+\sin y}{1-\sin y}}=\dfrac{1+\sin y}{\cos y}$ and $=\dfrac{\cos y}{1-\sin y}$.

28. $\dfrac{1+\tan x\cot 2x}{\cot x+\cot 2x}=\tan x.$ 29. $\dfrac{2\tan\frac{1}{2}A}{1+\tan^2\frac{1}{2}A}=\sin A.$

30. $\dfrac{\tan^2(45^\circ-x)-1}{\tan^2(45^\circ-x)+1}=-\sin 2x.$

31. $\dfrac{\sin 3\theta+\cos 3\theta}{\cos\theta-\sin\theta}=2\sin 2\theta+1.$

32. $\sin x+\sin 3x+\sin 5x+\sin 7x=4\sin 4x\cos 2x\cos x.$

33. $\dfrac{\sin 2\alpha-\sin\alpha}{1-\cos\alpha+\cos 2\alpha}=\tan\alpha.$

34. $\tan(x+y+z)=\dfrac{\tan x+\tan y+\tan z-\tan x\tan y\tan z}{1-\tan x\tan y-\tan y\tan z-\tan x\tan z}.$

35. $\dfrac{\cot\frac{1}{2}B+1}{\cot\frac{1}{2}B-1}=\dfrac{\cos B}{1-\sin B}.$ 36. $\dfrac{\sin 2\theta+\sin\theta}{\cos 2\theta+\cos\theta}=\tan\frac{3}{2}\theta.$

37. $\dfrac{\sin 3x-\cos 3x}{\sin x+\cos x}=2\sin 2x-1.$

38. $\dfrac{\tan(A+B)-\tan B}{1+\tan(A+B)\tan B}=\tan A.$

39. $\cos 37^\circ 30'-\cos 22^\circ 30'+\sin 7^\circ 30'=0.$

40. $\dfrac{\tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)}{\tan\left(\dfrac{\pi}{4}-\dfrac{x}{2}\right)}=\dfrac{\sec x+\tan x}{\sec x-\tan x}.$

41. $\dfrac{\sin 2n-\sin n}{\cos 2n-\cos n}=-\cot\frac{3}{2}n.$

42. $\tan\frac{1}{2}\theta+\cot\frac{1}{2}\theta=2\csc\theta.$

43. $\left(\frac{\tan \frac{1}{2} \phi + 1}{\tan \frac{1}{2} \phi - 1}\right)^2 = \frac{1 + \sin \phi}{1 - \sin \phi}.$

44. $(\sin \frac{1}{2} \alpha + \cos \frac{1}{2} \alpha)^2 = 1 + \sin \alpha.$

45. $(\sin \frac{1}{2} \alpha - \cos \frac{1}{2} \alpha)^2 = 1 - \sin \alpha.$

46. $\frac{1 + \sin x - \cos x}{1 + \sin x + \cos x} = \tan \frac{1}{2} x.$

Change each of the following to a monomial expression to which logarithms might be applied:

47. $\tan x + \cot x.$

48. $\tan A + \tan B.$

49. $\sin 32° + \sin 14°.$

50. $\cot \theta - \tan \theta'.$

51. $\frac{1 - \cos 2x}{1 + \cos 2x}.$

52. $\tan x \tan y - 1.$

53. $\tan 10° \tan 15° + 1.$

54. $\cot A \tan B - 1.$

55. $\frac{\tan x + \tan y}{\cot x + \cot y}.$

56. $\frac{\sin 19° + \sin 27°}{\cos 19° + \cos 27°}.$

57. If $\sin x = \frac{3}{5}$ and x is an angle in quadrant II, find $\cos \frac{1}{2} x$.

58. If $\cos \alpha = \frac{1}{3}$ and α is in quadrant IV, find $\cos (60° + \alpha)$. $\tan 2\alpha$. $\sin \frac{1}{2} \alpha$. $\cos \frac{1}{2} \alpha$.

59. If $\tan x = \frac{5}{12}$ and x is in quadrant III, find $\cos 2x$ and $\tan \frac{1}{2} x$.

60. If $\sin A = \frac{2}{3}$ and $\cos B = \frac{1}{3}$, $\angle A$ is in quadrant II, $\angle B$ is in quadrant I, find $\sin (A - B)$ and $\tan (A - B)$.

61. If $\sin x = \frac{4}{5}$ and $\tan y = -\frac{3}{4}$, $\angle x$ is in II, y is in II, find $\cos (x - y)$ and $\sin (x + \frac{1}{2} y)$.

62. If $\tan \frac{1}{2} \theta = \sqrt{2}$, find $\cos \theta$ and $\tan 2\theta$.

63. If $\sin \left(\frac{\pi}{2} + x\right) = \frac{3}{5}$, and x is $< 90°$, find $\tan x$.

64. If $\cos (\pi - x) = \frac{3}{4}$, and x is in quadrant II, find $\sin x$. $\sin \frac{1}{2} x$.

65. If $\sin (\pi + A) = -x$, and $\angle A$ is $< 90°$, find $\cos (\pi - 2A)$.

66. If $\tan\left(\frac{3\pi}{2}+\theta\right) = -m$, find $\sin\left(\frac{\pi}{2}-\theta\right)$.

67. If $\cos\left(\frac{\pi}{2}+x\right) = a$, find the values of $\sin 2x$ and $\cos 2x$.

68. Given $\sin \frac{1}{2} A = \frac{3}{5}$; find $\cos A$; $\tan A$; $\tan \frac{1}{2} A$; $\sin 2A$.

69. Given $\tan 2x = 3\frac{3}{7}$ and x an angle in the first quadrant find $\tan x$; $\cos x$; $\tan \frac{1}{2} x$; $\cos 2x$.

70. Solve example 69 if x is an angle in the second quadrant.

71. Given $\tan \frac{1}{2} \theta = \frac{1}{2}\sqrt{6}$ and θ an angle in the second quadrant. Find $\cos \theta$; $\tan \theta$; $\tan 2\theta$.

72. Prove: $1 + \cos a + \sin a = \sqrt{2(1 + \cos a)(1 + \sin a)}$.

(Square left member and take square root of result after reducing it.)

Prove: **73.** $\text{vers}\, 2x = 2 \sin^2 x$. **74.** $\text{covers}\, 2x = (\sin x - \cos x)^2$.

EQUATIONS

68. We will now consider the solution of equations involving some of the formulas of this chapter.

Ex. **1.** Solve $\tan x = \sin 2x$ for values of $x < 360°$.

Solution: $\frac{\sin x}{\cos x} = 2 \sin x \cos x$. $\therefore 2 \sin x \cos^2 x - \sin x = 0$.

$\therefore \sin x = 0$

and $x = 0, 180°$;

or $2 \cos^2 x = 1$, $\cos x = \pm \frac{1}{2}\sqrt{2}$,

$\therefore x = 45°, 135°, 225°, 315°$.

Ex. **2.** Solve $\sin \theta + \sin 2\theta = \sin 3\theta$ for values of $\theta < 360°$.

Solution: Write the equation, $\sin 3\theta - \sin \theta = \sin 2\theta$.

Collect the left member [by (20)], $2 \cos 2\theta \sin \theta = \sin 2\theta = 2 \sin \theta \cos \theta$.

$\therefore 2 \sin \theta = 0$

$\therefore \theta = 0°, 180°$;

or $\cos 2\theta - \cos \theta = 0$. $\therefore 2 \cos^2 \theta - 1 - \cos \theta = 0$.

$\therefore \cos \theta = 1$ or $-\frac{1}{2}$.

$\therefore \theta = 0°, 120°, 240°$.

Ex. 3. Solve, for values of $x < 360°$, $\sin \frac{1}{2} x + \cos x = 1$.

SOLUTION: $\sqrt{\frac{1 - \cos x}{2}} + \cos x = 1$. Transpose, square, collect,

$$2 \cos^2 x - 3 \cos x + 1 = 0. \quad \text{Hence, } x = 0°, 60°, 300°.$$

The pupil should verify each of the preceding solutions.

EXERCISE 37

odd

Find the values less than 360°, of the unknown angle in each equation, and verify:

1. $\sin x = \sin 2x$.
2. $\cos x = \sin 2x$.
3. $\tan 2x + \tan x = 0$.
4. $\sin 2x = 2 \cos x$.
5. $\cos \frac{1}{2} \theta = \sin \theta$.
6. $\tan \frac{1}{2} \theta = \sin \theta$.
7. $\sin \frac{1}{2} x + \cos \frac{1}{2} x = \sqrt{2}$.
8. $\sin \frac{1}{2} x - \cos \frac{1}{2} x = \frac{1}{2} \sqrt{2}$.
9. $\cos 2x = \sin x$.
10. $\cos 3x + \cos x = \sqrt{2} \cos 2x$
11. $\sin x + \cos 2x = 4 \sin^2 x$.
12. $4 \cos 2\theta + 3 \cos \theta = 1$.
13. $\cos 2\theta = 5 \sin^2 \theta - \cos^2 \theta$.
14. $2 \sin 2x - 1 = 2 \cos x - 2 \sin x$.
15. $\sin 2x = \sin x - 2 \cos x + 1$.
16. $\sin 2\theta = 2 \sin \theta - \cos \theta + 1$.
17. $\tan \theta + \cot 2\theta = \cot \theta$.
18. $\sin \theta + \sin 2\theta = 1 - \cos 2\theta$.
19. $6 \sin \theta = 5 - 4 \cos 2\theta$.
20. $\cos 5x - \cos 3x + \sin x = 0$.
21. $\cos 3x - \cos 2x + \cos x = 0$.
22. $\cos 2x + \cos x = -1$.
23. $\tan 2x + 2 \sin x = 0$.
24. $\tan 2\theta \tan \theta - 1 = 0$.
25. $\sin 4x - 2 \sin 2x = 0$.

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26. $\sin 2\theta + \sin\theta = \cos 2\theta + \cos\theta.$

27. $\sin x + \sin 2x + \sin 3x = 1 + \cos x + \cos 2x.$

28. $\sin 5x + \sin 3x + 2\cos x = 0.$

29. $\cos x - \cos 7x = \sin 4x.$

30. $\cos 3\theta + \sqrt{3}\cos 2\theta + \cos\theta = 0.$

31. $\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta = 0.$

32. $\sin\theta + \sin 2\theta + \sin 3\theta + \sin 4\theta = 0.$

33. $\sin 3x = 2\sin x.$ **34.** $\cos 3x + 8\cos^3 x = 0.$

35. $\sec^2 x + \csc^2 x = 5\frac{1}{3}.$

69. Simultaneous equations involving trigonometric functions may often be solved by simple algebraic methods, and many such equations require elaborate trigonometric reductions in their solution. Only a few of the simpler sort are given here.

Ex. 1. Solve for x and y: $\begin{cases} x\cos y = 5, \\ x\sin^2 y = 7\frac{1}{2}. \end{cases}$

Solution: To eliminate x, divide the first equation by the second.

$$\frac{\cos y}{\sin^2 y} = \frac{2}{3}. \quad \text{By solving, } y = 60^\circ \text{ or } 300^\circ.$$

Now, $$x = \frac{5}{\cos y}. \quad \therefore x = 10.$$

Ex. 2. Solve for x and θ: $\begin{cases} x\cos\left(\frac{\pi}{2} + \theta\right) = a, \\ x\sin\left(\frac{2\pi}{3} + \theta\right) = b. \end{cases}$

Solution: The first equation reduces to $-x\sin\theta = a$; $\therefore x = -\frac{a}{\sin\theta}$. The second equation becomes $x(\sin 120^\circ\cos\theta + \cos 120^\circ\sin\theta) = b.$

Substituting, $-\frac{a}{\sin\theta}\left[\frac{\sqrt{3}\cos\theta}{2} - \frac{\sin\theta}{2}\right] = b.$ $\quad\therefore -\frac{a\sqrt{3}}{2}\cot\theta + \frac{a}{2} = b.$

Whence, $\cot\theta = \frac{a - 2b}{a\sqrt{3}}.$ Inasmuch as a and b are considered known quantities, this equation gives the value of $\angle\theta$. Knowing θ, $\sin\theta$ is known. $\therefore x = -\frac{a}{\sin\theta}$ gives the value of x.

EXERCISE 38

Solve the following equations for the unknown quantities, x, y, $\angle\theta$:

1. $x\sin\theta=1$,
   $x\cos\theta=1$.

2. $x\cos\theta=2\sqrt{3}$,
   $x\sin\theta=-2$.

3. $8\cos\theta=y$,
   $y\tan^2\theta=12$.

4. $x\sin A+y\cos A=a$,
   $x\cos A+y\sin A=b$.

5. $x\cos\left(\theta-\frac{\pi}{2}\right)=2a$,
   $x\cos\left(\theta-\frac{\pi}{6}\right)=b$.

6. $y\sin(\theta+m)=a$,
   $y\sin(\theta-m)=b$.

7. Solve $\left\{\begin{matrix} T\sin\theta=\frac{1}{3} \\ T=\frac{4}{3}\sin\theta \end{matrix}\right\}$ for T and θ.

8. Solve $\left\{\begin{matrix} P\cos\theta=8 \\ P=10\cos\theta \end{matrix}\right\}$ for P and θ.

9. Solve $\left\{\begin{matrix} P\tan x=5 \\ P=5\sqrt{5}\cos x \end{matrix}\right\}$ for P and $\tan x$.

INVERSE TRIGONOMETRIC FUNCTIONS

70. Another symbol for an angle. We have seen that if $\angle A$ is acute, the equation $\sin A=\frac{a}{c}$ determines the magnitude of $\angle A$, for all real values that a and c may have. Thus if $\sin A=\frac{3}{7}$, $\angle A=$ that $\angle$ whose sine is $\frac{3}{7}$. The symbol abbreviating this description is "$\sin^{-1}\frac{3}{7}$." "$\mathrm{Sin}^{-1}\frac{a}{c}$" is read: "The $\angle$ whose sine is $\frac{a}{c}$," or "The inverse sine $\frac{a}{c}$." Also "$30°=\cot^{-1}\sqrt{3}$" is read: "30° is the angle whose cotangent is $\sqrt{3}$," and "$60°=\cos^{-1}\frac{1}{2}$" is read: "60° is the angle whose cosine is $\frac{1}{2}$." If we do not limit the angle to quadrant I, it is obvious that this new symbol is a symbol for either one of two angles, each less than 360°.

Thus, "$\sin^{-1}\frac{1}{2}$" is the symbol for 30° or 150°.

and "$\cos^{-1}(-\frac{1}{2}\sqrt{2})$" is the symbol for 135° or 225°.

EXERCISE 39

Read and explain the following equations, stating in which quadrant the angle concerned each time may be:

1. $A = \sin^{-1}\frac{5}{6}$.
2. $\theta = \tan^{-1}\frac{2}{3}$.
3. $\cos^{-1}\frac{4}{5} = \tan^{-1}\frac{3}{4}$.
4. $\tan^{-1}\frac{1}{7} = 2x$.
5. $2\sin^{-1}\frac{1}{2} = \cos^{-1}\frac{1}{2}$.
6. $\frac{1}{2}\tan^{-1}\sqrt{3} = \cot^{-1}\sqrt{3}$.
7. $\sec^{-1} m = 2\sin^{-1} n$.
8. $\csc^{-1}\frac{3}{2} = \frac{1}{2}\cos^{-1} x$.
9. $\sin(\cos^{-1}\frac{1}{2}) = \frac{1}{2}\sqrt{3}$.
10. $\cos^{-1}(-\frac{3}{4}) = \theta$.
11. $\cos(2\sin^{-1}\frac{1}{2}) = \frac{1}{2}$.
12. $\cos(2\sin^{-1}\frac{1}{3}) = \frac{7}{9}$.
13. $\sin(\sin^{-1}\frac{2}{11}) = \frac{2}{11}$.
14. $\cos^{-1}(\cos\theta) = \theta$.

Give the angles in each case, in degrees and in radians:

15. $\sin^{-1}\frac{1}{2}\sqrt{3}$.
16. $\tan^{-1}\frac{1}{3}\sqrt{3}$.
17. $\cos^{-1}(-\frac{1}{2})$.
18. $2\sin^{-1}(\pm\frac{1}{2})$.
19. $\sec^{-1}(\pm\sqrt{2})$.
20. $\frac{1}{3}\cos^{-1}(-1)$.
21. $\frac{1}{2}\tan^{-1}(\pm\sqrt{3})$.
22. $2\cot^{-1}(\pm 1)$.
23. $\frac{1}{5}\csc^{-1}(\pm 2)$.

Construct:

24. $\sin^{-1}\frac{2}{5}$.
25. $\cos^{-1}\frac{1}{3}$.
26. $\tan^{-1}(-2)$.
27. $\sin^{-1}(-\frac{3}{7})$.
28. $\cos^{-1}(-\frac{2}{3})$.
29. $\cot^{-1}\frac{3}{4}$.
30. $\sec^{-1}(-4)$.
31. $\frac{1}{2}\sin^{-1}\frac{1}{5}$.

NOTE. The symbol "$\sin^{-1} x$" is not the expression $(\sin x)^{-1}$, and they should not be confused, because they represent entirely different magnitudes.

$\sin^{-1} x$ is an angle; $(\sin x)^{-1}$ is a fraction.

The "-1" in the first, is only part of the symbol, in the second it is an exponent.

71. OPERATIONS WITH INVERSE FUNCTIONS

Ex. 1. Find $\sin(\tan^{-1}\frac{1}{3})$.

Let $\tan^{-1}\frac{1}{3} = A$. $\therefore \tan A = \frac{1}{3}$. $\therefore \sin A = \pm\frac{1}{10}\sqrt{10}$. (See § 44.)

$$\therefore \sin(\tan^{-1}\tfrac{1}{3}) = \pm\tfrac{1}{10}\sqrt{10}. \qquad \text{Q.E.F.}$$

Ex. 2. Find $\cos(2\cot^{-1} 2)$.

$1 - 2\sin^2$

Let $\cot^{-1} 2 = A$. $\therefore \cot A = 2$.

If $\cot A = 2$, $\sin A = \pm\frac{1}{5}\sqrt{5}$.

Now $\cos 2A = 2\overset{\text{cos}}{\cancel{\sin^2}} A - 1$.

$\therefore \cos 2A = 2(\frac{1}{5}) - 1 = +\frac{3}{5}$.

$$\therefore \cos(2\cot^{-1} 2) = -\tfrac{3}{5}. \qquad \text{Q.E.F.}$$

Ex. 3. Prove: $\sin(2\cos^{-1}x) = 2x\sqrt{1-x^2}$.

Proof: Let $\cos^{-1}x = A$. $\therefore \cos A = x$.

If $\cos A = x$, $\sin A = \sqrt{1-x^2}$. $\therefore \sin 2A = 2\sin A\cos A = 2x\sqrt{1-x^2}$.

$$\therefore \sin(2\cos^{-1}x) = 2x\sqrt{1-x^2}. \qquad \text{Q.E.D.}$$

Ex. 4. Derive a formula for $\tan(\tan^{-1}m - \tan^{-1}n)$.

Derivation: Let $\tan^{-1}m = A$, $\tan^{-1}n = B$. $\therefore \tan A = m$, $\tan B = n$.

$$\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A\tan B} = \frac{m-n}{1+mn}.$$

$$\therefore \tan(\tan^{-1}m - \tan^{-1}n) = \frac{m-n}{1+mn}. \qquad \text{Q.E.F.}$$

Ex. 5. Transform $\cos 2A = 2\cos^2 A - 1$ into a formula of inverse functions.

Let $\cos A = m$. $\therefore A = \cos^{-1}m$. $\therefore \cos(2\cos^{-1}m) = 2m^2 - 1$.

Ex. 6. Prove: $\tan^{-1}\frac{1}{3} - \tan^{-1}\frac{1}{5} = \tan^{-1}\frac{1}{8}$.

Proof: Let $\tan^{-1}\frac{1}{3} = A$, $\tan^{-1}\frac{1}{5} = B$, $\tan^{-1}\frac{1}{8} = C$.

Find the tangent of each member of this equation.

(a) $\tan(\tan^{-1}\frac{1}{3} - \tan^{-1}\frac{1}{5}) = \tan(A-B) = \dfrac{\tan A - \tan B}{1+\tan A\tan B}$

$$= \frac{\frac{1}{3}-\frac{1}{5}}{1+\frac{1}{3}\cdot\frac{1}{5}} = \frac{1}{8}.$$

(b) $\tan(\tan^{-1}\frac{1}{8}) = \frac{1}{8}$. $\therefore \tan^{-1}\frac{1}{3} - \tan^{-1}\frac{1}{5} = \tan^{-1}\frac{1}{8}$. **(Ax. 1).**

EXERCISE 40

Find:

1. $\sin(\cos^{-1}\frac{3}{5})$.
2. $\tan(\sec^{-1}10)$.
3. $\cos(\tan^{-1}x)$.
4. $\sec(\sin^{-1}.4)$.
5. $\cos(\cot^{-1}2)$.
6. $\cot(\csc^{-1}3)$.

Construct:

7. $\sin^{-1}\frac{1}{3}$.
8. $\cos^{-1}\frac{5}{8}$.
9. $\tan^{-1}\frac{7}{3}$.
10. $\cos^{-1}(-\frac{3}{7})$.

Find:

11. $\cos(2\sin^{-1}\frac{1}{4})$. **13.** $\sin(2\tan^{-1}\frac{3}{4})$. **15.** $\tan 2(\tan^{-1}2)$.

12. $\sin(2\cos^{-1}\frac{1}{2})$. **14.** $\sin(\frac{1}{2}\cos^{-1}\frac{1}{8})$. **16.** $\tan\frac{1}{2}[\cos^{-1}(-\frac{4}{5})]$

Prove:

17. $\cos(\sin^{-1}x)=\sqrt{1-x^2}$.

18. $\sin(\cos^{-1}a)=\sqrt{1-a^2}$.

19. $\tan(\sin^{-1}b)=\dfrac{b}{\sqrt{1-b^2}}$.

20. $\tan(\frac{1}{2}\cos^{-1}x)=\sqrt{\dfrac{1-x}{1+x}}$.

21. $\cos(2\cos^{-1}x)=2x^2-1$.

22. $\cos(2\sin^{-1}x)=1-2x^2$.

23. $\tan(2\tan^{-1}n)=\dfrac{2n}{1-n^2}$.

24. $\sin(2\tan^{-1}m)=\dfrac{2m}{1+m^2}$.

25. $\sin^{-1}(-x)=-\sin^{-1}x$.

26. $2\sin^{-1}a=\cos^{-1}(1-2a^2)$.

27. $\tan(\tan^{-1}x+\tan^{-1}y)=\dfrac{x+y}{1-xy}$.

28. $\cos(\cos^{-1}a+\cos^{-1}b)=ab-\sqrt{(1-a^2)(1-b^2)}$.

29. $\sin(\sin^{-1}m\pm\sin^{-1}n)=m\sqrt{1-n^2}\pm n\sqrt{1-m^2}$.

30. $\cos(\sin^{-1}x\pm\cos^{-1}y)=y\sqrt{1-x^2}\mp x\sqrt{1-y^2}$.

31. $\sin(\sin^{-1}a\pm\cos^{-1}b)=ab\pm\sqrt{(1-a^2)(1-b^2)}$.

Compute the value of:

32. $\sin(\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{4}{5})$.

33. $\tan(\sec^{-1}3-\csc^{-1}2)$.

34. $\cos\left(\tan^{-1}\frac{1}{2}+\sin^{-1}\dfrac{1}{\sqrt{10}}\right)$.

35. $\sin^{-1}m+\cos^{-1}m=90°$.

36. $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=45°$.

37. $\tan^{-1}a+\cot^{-1}a=\dfrac{\pi}{2}$.

38. $2\tan^{-1}\frac{1}{2}-\tan^{-1}\frac{1}{7}=\dfrac{\pi}{4}$.

39. $\tan^{-1}3+\tan^{-1}\frac{1}{3}=\dfrac{\pi}{2}$.

40. $\sin^{-1}\frac{3}{5}-\sin^{-1}\frac{5}{13}=\sin^{-1}\frac{16}{65}$.

41. $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{4}=45°-\tan^{-1}\frac{1}{13}$.

42. Prove that $x=y$ if $\sin^{-1}x+\sin^{-1}y=\pi$.

72. Equations. Often in the solution of trigonometric equations a value of a function is obtained which is possible, but which does not give the value of the angle without consultation of the natural tables. For example, $\sin\theta = \frac{3}{5}$. The value of θ can be determined numerically by use of the tables of natural functions. But it is customary to express θ by the symbol of inverse functions. Thus, if $\sin\theta = \frac{3}{5}$, $\theta = \sin^{-1}\frac{3}{5}$.

Ex. 1. Solve: $6\cos^2\theta + \cos\theta = 2$.

SOLUTION: Solving this quadratic equation, $\cos\theta = \frac{1}{2}$ or $-\frac{2}{3}$.
$\therefore \theta = 60°, 300°, \cos^{-1}(-\frac{2}{3})$.

Ex. 2. Solve for x, $\cos^{-1}x = 2\sin^{-1}x$.

SOLUTION: This is an equation in angles. Hence, we may take the cosine of each member of the equation.

$\therefore \cos(\cos^{-1}x) = \cos(2\sin^{-1}x)$. This reduces to, $x = \cos(2\sin^{-1}x)$. Now let $\sin^{-1}x = A$. $\therefore \sin A = x$. Substituting, $x = \cos 2A$.

$$\therefore x = 1 - 2\sin^2 A \text{ or } x = 1 - 2x^2.$$

Solving this quadratic, $x = \frac{1}{2}$ or -1.

The student should verify this result.

EXERCISE 41

Solve:

1. $10\sin^2\theta - \sin\theta = 2$.
2. $\tan^2 x + 4\tan x = 5$.
3. $3\cos^2 x + 2\cos x = 1$.
4. $\sec\theta + \tan\theta = 2$.
5. $\tan\theta + \cot\theta = 4$.
6. $\sec^2 x + \tan x = 7$.
7. $(4\sin^2 x - 3)(4\sin x - 3) = 0$
8. $\sin x + \cos x = 1$.
9. $4\cos 2\theta = \csc\theta$.
10. $\sin x + \sin 3x = 0$.
11. $\sin^{-1}x = 2\cos^{-1}x$.
12. $\sin^{-1}x = 2\tan^{-1}x$.
13. $\tan^{-1}x = 2\sin^{-1}x$.

RELATIONS AMONG THE SIDES AND THE FUNCTIONS OF THE ANGLES OF A TRIANGLE

73. Definition of s. Half the perimeter of a triangle is called s. That is, $a+b+c=2s$, the sides being a, b, c.

$$\therefore\ -a+b+c=2(s-a);\quad a-b+c=2(s-b);$$
$$a+b-c=2(s-c).$$

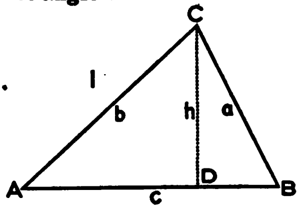
The advantage of this notation will become apparent in the solution of oblique triangles.

74. Law of Sines. In any triangle the ratio of any two sides is equal to the ratio of the sines of the opposite angles.

Given: $\triangle ABC$ whose sides are a, b, c.

Fig. I is an acute $\triangle$.

Fig. II is an obtuse $\triangle$.



To Prove: $\frac{a}{b}=\frac{\sin A}{\sin B}$; $\frac{b}{c}=\frac{\sin B}{\sin C}$; etc.

Proof: Draw the altitude CD in each $\triangle$.

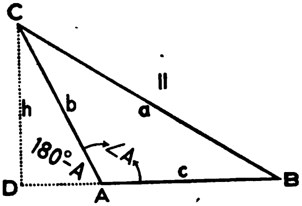


Fig. I. In $\triangle ACD$, $\frac{h}{b}=\sin A$. (1)

In $\triangle BCD$, $\frac{h}{a}=\sin B$. (2)

Dividing, $\frac{a}{b}=\frac{\sin A}{\sin B}$.

Fig. II. In $\triangle ACD$, $\frac{h}{b}=\sin(180°-A)=\sin A$.

In $\triangle BCD$, $\frac{h}{a}=\sin B$.

Dividing, $\frac{a}{b}=\frac{\sin A}{\sin B}$.

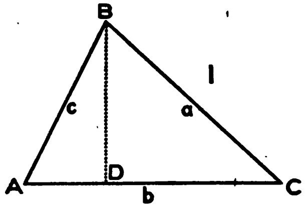
Hence, for any $\triangle$, $\frac{a}{b}=\frac{\sin A}{\sin B}$.

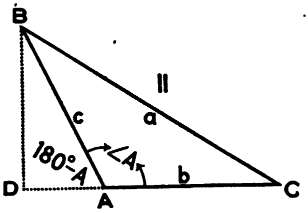
Likewise, $\frac{b}{c}=\frac{\sin B}{\sin C}$; and $\frac{a}{c}=\frac{\sin A}{\sin C}$.

These may be combined in a single formula,

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}.$$

75. Law of Cosines. In any triangle the square of one side is equal to the sum of the squares of the other two sides minus twice the product of these two sides times the cosine of their included angle.





Given: $\triangle ABC$ whose sides are a, b, c.

In Fig. I, a is opposite an acute angle.

In Fig. II, a is opposite an obtuse angle.

To Prove: $a^2=b^2+c^2-2bc\cos A$.

Proof: Draw the altitude from B, in both figures.

Fig. I. $a^2=b^2+c^2-2b\cdot AD$ (Geom. 346).

But $\frac{AD}{c}=\cos A$; $\therefore AD=c\cdot\cos A$.

Hence, $a^2=b^2+c^2-2\,b\cdot c\cdot\cos A$.

Fig. II. $a^2=b^2+c^2+2b\cdot AD$ (Geom. 345).

But $\frac{AD}{c}=\cos(180^\circ-A)=-\cos A$; $\therefore AD=-c\cdot\cos A$.

Hence, $a^2=b^2+c^2-2\,b\cdot c\cdot\cos A$.

Therefore, for any $\triangle$, $a^2=b^2+c^2-2\,bc\cos A$.

Likewise, $b^2=a^2+c^2-2\,ac\cos B$,

and $c^2=a^2+b^2-2\,ab\cos C$.

76. Problem. To derive a formula for the cosine of each angle of a triangle in terms of the sides of the triangle.

Formulas. From the equations of § 75,

$$\cos A=\frac{b^2+c^2-a^2}{2\,bc}.\qquad \cos B=\frac{a^2+c^2-b^2}{2\,ac}.\qquad \cos C=\frac{a^2+b^2-c^2}{2\,ab}.$$

77. Law of Tangents. **In any triangle the difference of any two sides is to their sum as the tangent of half the difference of the opposite angles is to the tangent of half their sum.**

Given: $\triangle ABC$, whose sides are a, b, c.

To Prove: $\dfrac{a-b}{a+b}=\dfrac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}$; $\dfrac{a-c}{a+c}=\dfrac{\tan\frac{1}{2}(A-C)}{\tan\frac{1}{2}(A+C)}$; etc.

Proof: $a:b=\sin A:\sin B$ (§ 74).

$$\therefore\ \frac{a-b}{a+b}=\frac{\sin A-\sin B}{\sin A+\sin B} \quad \text{(Division and Comp., Geom. 296).}$$

$$=\frac{2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)}{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)} \quad \text{[see formulas (20) and (19)]}$$

$$=\cot\tfrac{1}{2}(A+B)\tan\tfrac{1}{2}(A-B).$$

$$=\frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}.$$

Therefore, $$\frac{a-b}{a+b}=\frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}.$$

Also $$\frac{a-c}{a+c}=\frac{\tan\frac{1}{2}(A-C)}{\tan\frac{1}{2}(A+C)},$$

and $$\frac{b-c}{b+c}=\frac{\tan\frac{1}{2}(B-C)}{\tan\frac{1}{2}(B+C)}.$$

Note. If $b>a$ then $B>A$ (Geom. 122) and each numerator of the first formula will be negative. To avoid this, we may write it,

$$\frac{b-a}{b+a}=\frac{\tan\frac{1}{2}(B-A)}{\tan\frac{1}{2}(B+A)}.$$

Similarly the others are changed if $c>a$ or $c>b$.

Ex. 1. In $\triangle ABC$, if $\angle A = 45° 40'$, find $\frac{1}{2}(B + C)$.

SOLUTION: $A+B+C=180°$ (Geom. 110). $\therefore \frac{1}{2}A + \frac{1}{2}(B + C) = 90°$

But $\frac{1}{2}A = 22° 50'$

$\therefore \frac{1}{2}(B + C) = 67° 10'$.

Ex. 2. In $\triangle ABC$, if $\angle B = 80° 20'$ and $A - C = 35° 48'$ find A and C.

SOLUTION: We find $\frac{1}{2}(A + C) = 49° 50'$.
It is given, that $\frac{1}{2}(A - C) = 17° 54'$.
Adding, $A = 67° 44'$.
Subtracting, $C = 31° 56'$.

EXERCISE 42

In $\triangle ABC$,

1. If $B = 43° 18'$, find $\frac{1}{2}(A + C)$.

2. If $C = 61° 16'$ and $\frac{1}{2}(B - A) = 10° 28'$, find A and B.

3. If $B = 70° 32'$ and $\frac{1}{2}(C - A) = 25° 50'$, find A and C.

4. Show that $(s - a) + (s - b) + (s - c) = s$.

5. If $a = 25$, $b = 37$, $c = 48$, find s, $s - a$, $s - b$, $s - c$.

6. If $a = 7.18$, $b = 4.56$, $c = 9.42$, find s, $s - a$, $s - b$, $s - c$.

7. If $a = .4176$, $b = .8053$, $c = .7915$, find s, $s - a$, $s - b$, $s - c$.

8. Are the formulas of § 74 adapted to logarithmic computation?

9. Are the formulas of § 75 adapted to logarithmic computation?

10. Are the formulas of § 77 adapted to logarithmic computation?

11. If a, b, and A are known, which quantity can we find by the law of sines?

12. If a, b, c are known, which quantity can we find by the law of cosines?

13. If a, b, C are known, which quantities can we find by the law of tangents?

14. If a, A, B are known, which quantity can we find by the law of sines? Name another quantity that can also be found by the law of sines.

15. If two sides of a triangle and the angle opposite one of them are known, which law will find another part?

16. If two angles of a triangle and the side opposite one of them are known, which law will find another part?

17. If two sides of a triangle and the included angle are known, which law will find the two other angles? Which law will find the remaining side?

18. Is the law of sines true if $\angle C$ is a right angle? If $\angle B$ is a right angle?

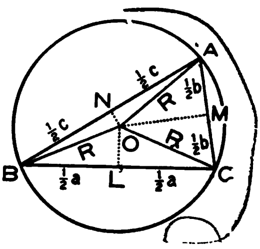
19. Is the law of cosines true if $\angle C$ is a right angle? If $\angle B$ is a right angle?

20. Is the law of tangents true if $\angle C$ is a right angle? If $\angle B$ is a right angle?

21. Prove, by use of the law of sines, that the bisector of an angle of a triangle divides the opposite side into segments proportional to the other two sides.

22. If R is the radius of the circle circumscribing a triangle ABC, prove that $2R = \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$. State this truth in words. (Draw ⊥s OL, OM, ON to the sides of the △s. These ⊥s bisect the sides and the central ∠s. (?) Hence,

$$\angle BOL = \angle COL = \angle A;\ \angle COM = \angle AOM = \angle B, \text{ etc.})$$



23. Discuss the law of sines, law of cosines, and law of tangents for the equilateral triangle.

78. Problem. To derive formulas for the sine and the cosine of half an angle of a triangle in terms of the sides of the triangle.

Given: $\triangle ABC$.

To Derive: Formulas for $\sin \frac{1}{2} A$ and $\cos \frac{1}{2} A$, etc.

Derivation: From (§ 61, 11),

$$2\sin^2 \tfrac{1}{2}A = 1 - \cos A.$$

From § 76,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

Substituting,

$$2\sin^2 \tfrac{1}{2}A = 1 - \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{2bc - b^2 - c^2 + a^2}{2bc}$$

$$= \frac{(a-b+c)(a+b-c)}{2bc}$$

$$= \frac{2(s-b)\cdot 2(s-c)}{2bc}.$$

$$\therefore \quad \sin^2 \tfrac{1}{2}A = \frac{(s-b)(s-c)}{bc},$$

$$\text{and} \quad \sin \tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{bc}}.$$

$$\text{Also,} \ \sin \tfrac{1}{2}B = \sqrt{\frac{(s-a)(s-c)}{ac}},$$

$$\text{and} \quad \sin \tfrac{1}{2}C = \sqrt{\frac{(s-a)(s-b)}{ab}}.$$

From (§ 61, 12),

$$2\cos^2 \tfrac{1}{2}A = 1 + \cos A.$$

From § 76,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

Substituting,

$$2\cos^2 \tfrac{1}{2}A = 1 + \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{2bc + b^2 + c^2 - a^2}{2bc}$$

$$= \frac{(b+c+a)(b+c-a)}{2bc}$$

$$= \frac{2(s)\cdot 2(s-a)}{2bc}.$$

$$\therefore \quad \cos^2 \tfrac{1}{2}A = \frac{s(s-a)}{bc},$$

$$\text{and} \ \cos \tfrac{1}{2}A = \sqrt{\frac{s(s-a)}{bc}}.$$

$$\text{Also,} \ \cos \tfrac{1}{2}B = \sqrt{\frac{s(s-b)}{ac}},$$

$$\text{and} \ \cos \tfrac{1}{2}C = \sqrt{\frac{s(s-c)}{ab}}.$$

79. Problem. **To derive a formula for the tangent of half an angle of a triangle in terms of the sides of the triangle.**

Derivation:

$$\tan \tfrac{1}{2}A = \frac{\sin \tfrac{1}{2}A}{\cos \tfrac{1}{2}A} = \frac{\sqrt{\dfrac{(s-b)(s-c)}{bc}}}{\sqrt{\dfrac{s(s-a)}{bc}}} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

$$\therefore \tan\tfrac{1}{2}A = \frac{1}{s-a}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}},$$

$$\text{and } \tan\tfrac{1}{2}B = \frac{1}{s-b}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}},$$

$$\text{and } \tan\tfrac{1}{2}C = \frac{1}{s-c}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

EXERCISE 43

1. Derive from the law of tangents, the formulas:

$$\tan\tfrac{1}{2}(A-B) = \frac{a-b}{a+b}\cdot\tan\tfrac{1}{2}(A+B);$$

and

$$\tan\tfrac{1}{2}(A-B) = \frac{a-b}{a+b}\cdot\cot\tfrac{1}{2}C.$$

2. Prove that in any triangle ABC, $c = a\cos B + b\cos A$. (Use figures of § 74. $c = BD + AD$ or $c = BD - AD$, etc.)

3. Prove that in any triangle ABC, $a = b\cos C + c\cos B$. Also $b = a\cos C + c\cos A$.

4. Prove that in any triangle ABC,

$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}.$$

(Use the formula $\sin 2x = 2\sin x\cos x$. $\therefore \sin A = 2\sin\tfrac{1}{2}A\cos\tfrac{1}{2}A$, etc.)

5. If the sides of a triangle are known and an angle is desired, the right members of the formulas of §§ 78 and 79 are known. To find $\tfrac{1}{2}A$ by either formula of § 78 requires the finding of how many logarithms? How many logarithms are required if one uses the formula of § 79? To find $\tfrac{1}{2}A$, $\tfrac{1}{2}B$, and $\tfrac{1}{2}C$ how many logarithms must be found in the formulas of § 78? In the formulas of § 79?

6. Why are all the radicals of §§ 78 and 79 necessarily preceded by a positive sign?

7. Discuss the formulas of §§ 78 and 79 for the equilateral triangle.

CHAPTER V

SOLUTION OF OBLIQUE TRIANGLES

80. In order that a triangle may be constructed, three of its parts must be known and one of these three must be a side. Thus, the complete solution of oblique triangles may be subdivided into four cases, in which are given:

I. **One side and two angles.**
II. **Two sides and the included angle.**
III. **Three sides.**
IV. **Two sides and an angle opposite one of them.**

It is obvious that a triangle is not determined unless three independent elements are given. The three angles are not independent, because their sum is constant.

The first and fourth of these cases employ the law of sines.
The second employs the law of tangents.
The third employs formulas $\tan \frac{1}{2} A$, $\tan \frac{1}{2} B$, $\tan \frac{1}{2} C$ (§ 79).

CASE I. ONE SIDE AND TWO ANGLES

81. Ex. Solve the triangle ABC, if $a = 52.78$, $A = 37° 41' 15''$, $B = 77° 29' 40''$.

SOLUTION: $C = 180° - (A + B) = 64° 49' 5''$.

To find b and c we use the law of sines, writing the formula so that the unknown letter appears first.

$$\frac{b}{a} = \frac{\sin B}{\sin A}. \qquad \text{Also,} \quad \frac{c}{a} = \frac{\sin C}{\sin A}.$$

$$\therefore b = \frac{a \sin B}{\sin A}. \qquad \therefore c = \frac{a \sin C}{\sin A}.$$

The right-hand members of these equations being composed of known quantities, we may compute their values by logarithms, thus finding the values of b and c.

SCHEME

$$\begin{aligned} \log a &= 1.72247 \\ \log \sin B &= 9.98957 - 10 \\ \operatorname{colog} \sin A &= 0.21371 \\ \log b &= 1.92575 \\ \therefore\ b &= 84.285. \end{aligned}$$

SCHEME

$$\begin{aligned} \log a &= 1.72247 \\ \log \sin C &= 9.95663 - 10 \\ \operatorname{colog} \sin A &= 0.21371 \\ \log c &= 1.89281 \\ \therefore\ c &= 78.128. \end{aligned}$$

VERIFICATION. By law of sines, $\frac{b}{c} = \frac{\sin B}{\sin C}$. That is, if the above results are correct, $\frac{84.285}{78.128}$ should equal $\frac{.97628}{.90496}$.

Each of these fractions reduces to 1.0788+. Hence the three values we have found are correct.

Ordinary division and natural functions were used in this check to make its test more accurate. A check permitting the use of logarithms of numbers whose logarithms have not been found in the solution, and a very accurate check, is the law of tangents: $\frac{b-c}{b+c} = \frac{\tan \frac{1}{2}(B-C)}{\tan \frac{1}{2}(B+C)}$.

NOTES. I. This case is always possible provided the sum of the given angles is $< 180°$.

II. In finding the log sin A, if A is an obtuse angle, reduce to an acute angle by § 45. Log sin 130° 10′ = log sin 49° 50′.

EXERCISE 44

Find the remaining parts of each of the following triangles:

1. $a = 2.901$, $B = 69° 40'$, $C = 33° 15'$.
2. $c = 663.986$, $A = 79° 59'$, $B = 44° 41'$.
3. $b = 90.61$, $A = 40° 37'$, $C = 40° 1'$.
4. $a = 209.3$, $B = 21° 6'$, $C = 123° 39'$.
5. $a = 37.952$, $B = 63° 41' 36''$, $C = 51° 19' 50''$.
6. $c = 1562.025$, $A = 104° 31' 9''$, $B = 32° 2' 52''$.
7. $b = 41.554$, $A = 118° 55' 49''$, $C = 45° 41' 35''$.
8. $a = .075632$, $A = 54° 12' 18''$, $B = 40° 30' 14''$.
9. $b = 740.021$, $A = 32° 25' 46''$, $B = 37° 23' 25''$.
10. $B = 44° 1' 28''$, $A = 97° 44' 20''$, $c = .08716$.

CASE II. TWO SIDES AND THE INCLUDED ANGLE

82. Ex. Solve the triangle ABC, if $A = 35° \; 17' \; 28''$, $b = .941$, $c = 1.256$.

Solution: To find $\angle B$ and C we use the law of tangents.

Because $c > b$, $\angle C > \angle B$ (Geom. 122). Hence, the formula is written, — the unknown term appearing first:

$$\frac{\tan \frac{1}{2}(C-B)}{\tan \frac{1}{2}(C+B)} = \frac{c-b}{c+b}. \quad \therefore \tan \tfrac{1}{2}(C-B) = \frac{(c-b)\tan \frac{1}{2}(C+B)}{c+b}.$$

$$\begin{aligned} c + b &= 2.197 \\ c - b &= .315 \\ \tfrac{1}{2}(C+B) &= 72° \; 21' \; 16'' \\ \tfrac{1}{2}(C-B) &= \underline{24° \; 15' \; 48''} \\ C &= 96° \; 37' \; 4'' \\ B &= 48° \; 5' \; 28''. \end{aligned}$$

$$\overbrace{\begin{aligned} \log(c-b) &= 9.49831 - 10 \\ \log \tan \tfrac{1}{2}(C+B) &= 0.49745 \\ \text{colog}\,(c+b) &= \underline{9.65817 - 10} \\ \log \tan \tfrac{1}{2}(C-B) &= 9.65393 - 10 \\ \therefore \tfrac{1}{2}(C-B) &= 24° \; 15' \; 48''. \end{aligned}}^{\text{Scheme}}$$

Verification. By the law of sines, $\frac{b}{c} = \frac{\sin B}{\sin C}$. We have found the logarithm of none of these four quantities in the solution, and hence we may use logarithms in this check, or not, as we please.

If the above values are correct, $\frac{.941}{1.256}$ is equal to $\frac{\sin 48° \; 5' \; 28''}{\sin 96° \; 37' \; 4''}$.

Each of these fractions is equal to .7492, hence our results are correct.

To find a, we may use

$$\frac{a}{b} = \frac{\sin A}{\sin B}, \quad \therefore a = \frac{b \sin A}{\sin B}.$$

$$\overbrace{\begin{aligned} \log b &= 9.97359 - 10 \\ \log \sin A &= 9.76172 - 10 \\ \text{colog} \sin B &= \underline{0.12831} \\ \log a &= 9.86362 - 10 \\ \therefore a &= .7305. \end{aligned}}^{\text{Scheme}}$$

Note. This case is always possible.

EXERCISE 45

Find the remaining parts in each of the following triangles:

1. $a = 24$,
   $b = 17$,
   $C = 37° \; 28'$.
2. $b = 79$,
   $c = 107$,
   $A = 71° \; 35'$.
3. $c = 408$,
   $a = 631$,
   $B = 63° \; 41'$.

4. $B = 81° 10' 30''$,
$a = 35.7$,
$c = 70.6$.

5. $a = .12765$,
$b = .09378$,
$C = 44° 27' 16''$

6. $a = 31.705$,
$b = 42.086$,
$C = 21° 19' 48''$.

7. $A = 61° 29' 36''$,
$b = 478.56$,
$c = 639.28$.

8. $a = 389.76$,
$c = 410.95$,
$B = 71° 41' 38''$.

9. $a = .007632$,
$b = .004793$,
$C = 102° 3' 42''$.

10. $b = 417.328$, $c = 706.305$, $A = 123° 17' 52''$.

CASE III. THREE SIDES

83. Ex. Solve $\triangle ABC$, if $a = .0375$, $b = .0878$, $c = .0711$.

Solution: To find the three angles, we use the formulas of § 79.

$$\tan \tfrac{1}{2} A = \frac{1}{s-a}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}};$$

$$\tan \tfrac{1}{2} B = \frac{1}{s-b}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}};\quad \tan \tfrac{1}{2} C = \text{etc.}$$

	Scheme	Scheme for ∠A
$s = .0982$	$\log (s-a) = 8.78319 - 10$	log rad. $= 8.12054 - 10$
$s - a = .0607$	$\log (s-b) = 8.01703 - 10$	$\text{colog}\,(s-a) = 1.21681$
$s - b = .0104$	$\log (s-c) = 8.43297 - 10$	$\log \tan \tfrac{1}{2} A = 9.33735 - 10$
$s - c = .0271$	$\text{colog}\, s = 1.00789$	$\therefore \tfrac{1}{2} A = 12° 16' 4''$.
$s = .0982$	$2)16.24108 - 20$	$\therefore A = 24° 32' 8''$.
(check)	log rad. $= 8.12054 - 10$	

Scheme for ∠B	Scheme for ∠C
log rad. $= 8.12054 - 10$	log rad. $= 8.12054 - 10$
$\text{colog}\,(s-b) = 1.98297$	$\text{colog}\,(s-c) = 1.56703$
$\log \tan \tfrac{1}{2} B = 10.10351 - 10$	$\log \tan \tfrac{1}{2} C = 9.68757 - 10$
$\therefore \tfrac{1}{2} B = 51° 45' 50.7''$	$\therefore \tfrac{1}{2} C = 25° 58' 5.6''$
$B = 103° 31' 41.4''$	$\therefore C = 51° 56' 11.2''$

Verification. $A = 24° 32' 8''$
$B = 103° 31' 41.4''$
$C = 51° 56' 11.2''$
$\text{Sum} = 180° 0' 0.6'$

Note. This case is always possible provided the sum of every pair of sides is > the third side.

EXERCISE 46

Find the remaining parts in each of the following triangles:

1. $a = 53$,
$b = 67$,
$c = 72$.

2. $a = 6.5$,
$b = 7.36$,
$c = 9.14$.

3. $a = 48.6$,
$b = 34.7$,
$c = 38.35$.

4. $a = 128.4$,
$b = 144.4$,
$c = 160.8$.

5. $a = .0375$,
$b = .04185$,
$c = .03495$.

6. $a = 431.76$,
$b = 609.05$,
$c = 710.89$.

7. $a = .34678$,
$b = .91704$,
$c = .80006$.

8. $a = 1234$,
$b = 2345$,
$c = 3079$.

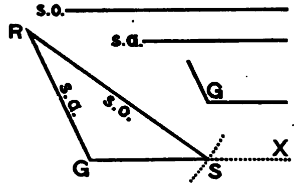
9. $a = 425.667$,
$b = 526.224$,
$c = 603.777$.

10. $a = 7.23246$, $b = 4.4818$, $c = 8.5086$.

CASE IV. TWO SIDES AND AN ANGLE OPPOSITE ONE OF THEM

That is, **one angle, a side adjoining it, and a side opposite it.**

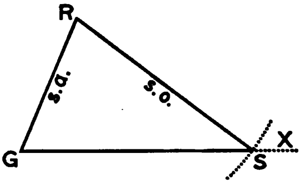
84. This is called **the ambiguous case**, because under certain conditions there may be two triangles each containing the given parts. Under other conditions there may be no triangle.



Discussion: Let us denote the given angle by "G," the side adjacent to $\angle G$ by "*s.a.*" the side opposite $\angle G$ by "*s.o.*"

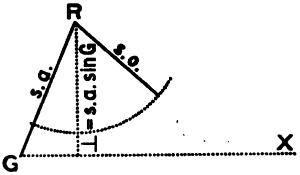
1. **If $\angle G$ is obtuse** or right, the triangle can be constructed and solved, if *s.o.* > *s.a.*

It is impossible if *s.o.* = *s.a.*, or if *s.o.* < *s.a.* (Geom. 123).



2. **If $\angle G$ is acute,** the triangle can be constructed and solved, if *s.o.* $>$ *s.a.* or if *s.o.* $=$ *s.a.*

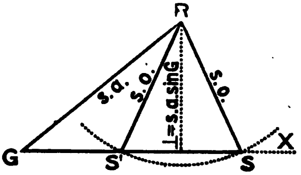
But if *s.o.* $<$ *s.a.* the solution depends upon the length of the $\perp$ from vertex *R* to the base *GX*. **This $\perp =$ *s.a.* $\times$ sin *G*.**



(1) If *s.o.* $<$ the $\perp$, there can be no $\triangle$.

(2) If *s.o.* $=$ the $\perp$, there is one right $\triangle$.

(3) If *s.o.* $>$ the $\perp$, there are two $\triangle$.



Each of the $\triangle$ *GRS* and *GRS'* contains the given parts.

Ex. 1. **Discuss the triangle *ABC*, if $a = 160$, $b = 100$, $A = 40°$.**

Discussion. $\angle A$ is acute and *s.o.* $>$ *s.a.*
$\therefore$ there is one $\triangle$ and only one.

Ex. 2. Discuss the triangle *ABC*, if $b=50$, $c=80$, $B=110°$.

Discussion. $\angle B$ is obtuse and *s.o.* $<$ *s.a.* $\therefore$ the $\triangle$ is impossible.

Ex. 3. Discuss triangle *ABC*, if $a = 160$, $b = 120$, $B = 40°$.

Discussion. $\angle A$ is acute and *s.o.* $<$ *s.a.* Hence we find the $\perp$. $\perp = 160 \cdot \sin 40° = 102.84$. $\therefore$ *s.o.* $>$ the $\perp$ and there are two $\triangle$.

Ex. 4. Discuss the triangle *ABC*, if $a=20$, $c=16$, $C=65°$.

Discussion. The $\perp = 20 \cdot \sin 65° = 18.13$. $\therefore$ the $\triangle$ is impossible.

Note. The length of the $\perp$ can be found either by natural functions or by logarithms.

Ex. 5. In discussing $\triangle$ *GRS*, in which $\angle G = 50°$, $s = 5.37$, $g = 4.68$, would you find the $\perp$? Why? What is true if $\perp = 4.36$? What is true if $\perp = 4.9$? If $\perp = 5.5$?

EXERCISE 47

Discuss each of the following triangles:

1. $a = 40$,
$b = 30$,
$A = 30°$.

2. $a = 40$,
$b = 50$,
$A = 100°$.

3. $a = 150$,
$b = 180$,
$A = 30°$.

4. $a = 70$,
$c = 35$,
$C = 30°$.

5. $b = 40$,
$c = 40$,
$B = 70°$.

6. $a = 72$,
$b = 160$,
$A = 30°$.

7. $a = 50$,
$b = 44$,
$A = 125°$.

8. $b = 76$,
$c = 51$,
$C = 30°$.

9. $a = 93$,
$b = 45$,
$A = 75°$.

10. $a = 9.7$,
$c = 7.5$,
$C = 14°$.

11. $a = 17.5$,
$b = 15.6$,
$B = 113°$.

12. $a = 446$,
$b = 288$,
$B = 44° 30'$.

13. $c = 21.3$,
$b = 49.598$,
$C = 25° 26'$.

14. $b = .0956$,
$c = .1016$,
$B = 64° 8'$.

15. $a = 3.509$,
$c = 3.814$,
$C = 93° 17'$.

85. Ex. 1. Solve the triangle ABC if $a = 6.315$, $c = 21.45$, $C = 130° 40' 18''$.

Discussion. $\angle C$ is obtuse and $c > a$.
$\therefore$ there is one solution and only one.

Solution. To find $\angle A$ we use the law of sines. Writing the formula so that the unknown term appears first: $\frac{\sin A}{\sin C} = \frac{a}{c}$.

$$\therefore \sin A = \frac{a \sin C}{c}.$$

SCHEME

$\log a \qquad = 0.80037$
$\log \sin C = 9.87993 - 10$
$\text{colog } c \quad = 8.66857 - 10$
$\log \sin A = 9.34887 - 10$
$\therefore A = 12° 54' 9''$.

$\angle B = 180° - (A + C) = 36° 25' 33''$.
To find b we use the law of sines.

$$b = \frac{c \sin B}{\sin C}.$$

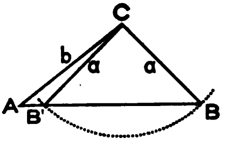
Whence, $b = 16.793$.

Verification. A good check is $\frac{a}{b}=\frac{\sin A}{\sin B}$, but one more accurate, though more laborious, is $\frac{a-b}{a+b}=\frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}$.

2. Ex. 2. Solve the triangle ABC, if $a=871.6$, $b=978.7$, $A=38° 14' 12''$.

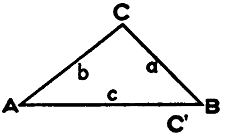
Discussion. $\angle A$ is acute, $a<b$.

$\perp = b \sin A = 605.7$. $\therefore$ there are two solutions.

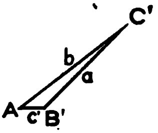


Solutions: Draw a triangle having the given angle at the lower left-hand vertex, and the unknown side the base, showing the two triangles ABC and $AB'C$. Draw these $\triangle$ separately. In the first figure, $\angle B = \angle CB'B$ (Geom. 55). $\angle AB'C = 180° - \angle CB'B$ (Geom. 44).

$\therefore \angle AB'C = 180° - \angle B$. That is, in the separate triangles $\angle B' = 180° - \angle B$.



The same truth may be shown another way: $\sin(180° - B) = \sin B$, thus if we find $\sin B$, we may, from it, find both the angles B and B', one being acute, the other its supplement. In other words, if an angle of a triangle is determined by its sine, the angle may have two values, one in the first quadrant, the other in the second.



To find $\angle B$ we use the law of sines, $\sin B = \frac{b \sin A}{a}$.

$\log b = 2.99065$

$\log \sin A = 9.79163 - 10$

$\operatorname{colog} a = 7.05968 - 10$

$\log \sin B = 9.84196 - 10$

$\therefore B = 44° \; 1' \; 28''$, in $\triangle ABC$.

And $B' = 135° \; 58' \; 32''$, in $\triangle AB'C'$.

In $\triangle ABC, \angle C = 180° - (A+B) = 97° 44' 20''$.

In $\triangle AB'C', \angle C' = 180° - (A+B') = 5° 47' 16''$.

To find c and c' we use the law of sines:

$$c=\frac{a \sin C}{\sin A};\quad c'=\frac{a \sin C}{\sin A}.$$

From these formulas the values found are: $c = 1395.4$; $c' = 142.02$.

These results may be verified for each $\triangle$, as in Ex. 1.

adj – [illegible]

EXERCISE 48

Find the remaining parts in each of the following triangles:

1. $a = 420$,
$b = 242.5$,
$A = 21° 31'$.

2. $b = 132$,
$a = 156.3$,
$B = 57° 37' 17''$.

3. $b = 45.21$,
$c = 50.3$,
$B = 40° 32' 7''$.

4. $a = 7.4728$,
$c = 15.768$,
$C = 114° 29' 51''$.

5. $b = 278.54$,
$c = 391.9432$,
$B = 45° 17' 20''$.

6. $a = .0929$,
$c = .0757$,
$C = 65° 38' 20''$.

7. $b = 1452$,
$c = 4602$,
$C = 36° 53' 2''$.

8. $b = 0.27096$,
$a = 0.32165$,
$B = 52° 24' 24''$.

9. $b = 5143.4$,
$c = 4795.56$,
$C = 72° 53' 38''$.

10. $a = 347.51$,
$c = 340.4$,
$C = 100° 28' 46''$.

11. $a = 365.24$,
$c = 346.45$,
$C = 62° 17' 48''$.

12. $a = 24.235$,
$c = 24.9165$,
$A = 76° 33' 50''$.

EXERCISE 49

MISCELLANEOUS EXERCISE

Find the remaining parts in each of the following triangles:

1. $a = 76.314$,
$B = 81° 43' 17''$,
$C = 27° 15' 45''$.

2. $b = 43.97$,
$c = 38.41$,
$A = 43° 28' 18''$.

3. $a = 1.513$,
$b = 2.176$,
$c = 2.097$.

4. $a = 18.476$,
$b = 9.511$,
$A = 70° 48' 25''$.

5. $a = 87.53$,
$b = 91.08$,
$c = 104.37$.

6. $a = 428.6$,
$b = 302.7$,
$C = 38° 47' 24''$.

7. $b = 28.119$,
$c = 25.842$,
$C = 38° 18' 29''$.

8. $b = 607$,
$A = 38° 27' 41''$,
$C = 77° 43' 8''$.

9. $a = .2176$,
$b = .4675$,
$c = .5039$.

10. $a = 3.147$,
$c = 3.886$,
$A = 62° 17' 9''$.

11. $a = 938.7$,
$B = 47° 28' 52''$,
$C = 18° 37''$.

12. $a = 17.631$,
$b = 13.914$,
$B = 28° 38' 48''$.

[illegible]

13. The pages of a book are 5 × 7 in. A diagonal is drawn across a page. Find the angles it forms with the edges of the page.

14. A parallelogram has two adjacent sides 23 and 40; the angle between them is 108°. Find the lengths of the diagonals.

15. My eye is 12 ft. from one end of a 10-ft. pole and 17 ft. from the other end. Find the angle at my eye subtended by the pole.

16. A field in the form of a triangle has one side 63 rd. in length and the angles at the ends of this side are 43° and 76°. Find the other sides.

17. The diagonals of a parallelogram are 66 and 84; the angle between them is 37° 14′. Find the sides of the parallelogram.

18. Two trains start at the same time from the same station and upon straight tracks diverging at an angle of 67° 45′. If the one runs 32 mi. an hour and the second 46 mi. an hour, how far apart are they at the end of three hours?

19. If the trains in example 18 are 125 mi. apart in 3 hr., at what angle do the railroads intersect?

20. At a certain point on a level with the base of a tower, I observe the angle of elevation of its top to be 18° 25′. I proceed directly toward the base 75 yd., and the angle of elevation of the top of the tower is now 31° 16′. Find the height of the tower.

21. The parallel sides of a trapezoid are 80 ft. and 133 ft. The angles at the ends of the latter side are 61° 18′ and 46° 27′. Find the lengths of the non-parallel sides.

22. What is the length of the shadow of a tree 132 ft. high, when the angle of elevation of the sun is 36° 21′?

23. If a chord 73.6 in. long subtends an arc of 116° 27′, what is the radius of the circle?

24. How far from the foot of a steeple 196 ft. high should one stand so that the angle of elevation of its top is 23° 15′?

25. Two sides of a parallelogram are 71 and 128 and the less diagonal is 146. Find the other.

26. Denote my position by *A* and the position of an inaccessible object by *B*. I measure a base $AC = 134$ yd., and the angles $BAC = 73° \, 14'$ and $BCA = 49° \, 10'$. Find the distance *AB*.

27. What is the visual angle of an object 12 ft. long, one end being 15 ft., and the other 8 ft., from the eye?

28. The angle between the legs of a pair of compasses is 72° 50′ and they are each 5.5 in. long. How far apart are the points?

29. There are three cities situated as follows: *A* is 215 mi. due south of *B*; *C* and *B* are 178 mi. apart; *C* is 196 mi. from *A*. In what direction is *C* from *A*? (Two admissible positions.)

30. I observe that the top of a church 64 ft. high has an angle of elevation of 19° 28′ 15″, and the angle of elevation of the top of the spire from the same point of observation is 37° 20′ 40″. What is the height of the top of the spire?

31. One of the equal sides of an isosceles triangle is 25.46 and one of the equal angles is 25° 46′. Find the third side and the altitude upon it.

32. A house and barn stand in a level plane. Viewed from the top of the house the angles of depression of the top and bottom of the barn are 18° and 53° respectively. The height of the house is 40 ft. Find the height of the barn.

33. Two stations *A* and *B* are inaccessible from *C*. It is known that $AB = 436$ yd., $\angle CAB = 53° \, 7'$, $\angle CBA = 70° \, 15'$. Find *AC* and *BC*.

34. A chord 4.063 in. long in a circle whose diameter is 25.95 in. long subtends a central angle. Find the magnitude of this angle.

35. At a point in a straight level road leading to a monument, I observe the angle of elevation of the top of the monument to be $\theta°$. I proceed n yards toward the monument and observe the angle of elevation of its top to be $\phi°$. Show that the height of the monument $= \dfrac{n \tan \theta \tan \phi}{\tan \phi - \tan \theta}$ yards.

36. At a point on the opposite bank of a river from a building, the angle of elevation of the top of the building is 60°. The observer withdraws 75 yd. from the bank in a horizontal plane, and notes that the angle of elevation of the top of the building is 30°. Find the height of the building and the width of the river.

37. When the sun is 40° high it is noticed that the length of the shadow of a tower is 185 ft., and the length of the shadow of a vertical flagpole on its top is 48 ft. Find the height of the tower and the length of the pole.

38. The sides of a parallelogram are 7.06 and 11.75 and an angle is 66° 45′. Find its altitudes.

AREA

86. Problem. To derive formulas for the area of a triangle in terms of its sides and the sines of its angles.

Derivation. I. Denote the area of a △ by Δ. In △ ABC draw altitude AD.

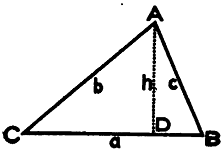
Now area △ $ABC = \frac{1}{2}a \cdot h$ (Geom. 378).

But $h = b \cdot \sin C$.

$\therefore \Delta = \frac{1}{2} a \cdot b \cdot \sin C$.

Likewise $\Delta = \frac{1}{2} b \cdot c \cdot \sin A$;

and $\Delta = \frac{1}{2} a \cdot c \cdot \sin B$.



II. By the law of sines, $b = \dfrac{a \sin B}{\sin A}$.

Substituting in the first formula of I, $\Delta = \frac{1}{2} a \cdot \dfrac{a \sin B}{\sin A} \cdot \sin C$.

$$\therefore \Delta = \frac{a^2 \sin B \sin C}{2 \sin A}.$$

Likewise $\Delta = \dfrac{b^2 \sin A \sin C}{2 \sin B}$; and $\Delta = \dfrac{c^2 \sin A \sin B}{2 \sin C}$.

III. By the formula (5) of § 59, $\sin 2x = 2 \sin x \cos x$.

Let $2x = A$, $x = \frac{1}{2}A$.

$$\therefore \sin A = 2 \sin \tfrac{1}{2} A \cos \tfrac{1}{2} A = 2\sqrt{\frac{(s-b)(s-c)}{bc}} \cdot \sqrt{\frac{s(s-a)}{bc}}.$$

$\therefore \sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}.$ Substituting in the second formula of I, $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$.

Ex. Find the area of ΔABC, if $b = 47.3$, $c = 21.8$, $A = 20° 25'$.

Solution: The formula is

$\Delta = \frac{1}{2} bc \sin A.$

Scheme

$$\begin{aligned}
\text{colog } 2 &= 9.69897 - 10\\
\log b &= 1.67486\\
\log c &= 1.33846\\
\log \sin A &= \underline{9.54263 - 10}\\
\log \Delta &= 2.25492\\
\therefore \Delta &= 179.85.
\end{aligned}$$

Note. The above formulas will apply to every case except the ambiguous case (Case IV, § 84). In order to find the area of a triangle in this case, it is necessary first to find the remaining angles.

EXERCISE 50

1. Interpret the first group of formulas of § 86 as a theorem for the area of a triangle.

2. Interpret the second group of formulas of § 86 as a theorem for the area of a triangle.

Find the area of each of the following triangles:

3. $a = 75.3$,
$b = 42.8$,
$C = 71° 28' 14''$.

4. $A = 19° 45'$,
$B = 83° 17'$,
$c = 7.128$.

5. $a = 92.43$,
$b = 73.48$,
$c = 60.77$.

6. $A = 72° 48'$,
$a = 12.391$,
$b = 9.417$.

7. $a = .915$,
$B = 44° 44' 32''$,
$C = 100° 41' 48''$.

8. $b = .0275$,
$c = .0198$,
$A = 60° 50' 28''$.

9. $a = 1.378$,
$b = 3.125$,
$c = 2.973$.

10. $A = 40° 20' 36''$,
$a = 17.953$,
$c = 21.43$.

11. $a = 5.3881$,
$c = 9.0774$,
$B = 73° 14' 40''$.

12. $a = 9.0705$,
$b = 8.4046$,
$c = 7.9555$.

13. Prove that the area of a parallelogram is equal to the product of two adjoining sides times the sine of their included angle.

14. Prove that the area of any quadrilateral is equal to half the product of the diagonals multiplied by the sine of either angle at their intersection.

15. Find the area of a parallelogram if its sides are 56 and 95, and their included angle is 62° 30′.

16. Find the area of a parallelogram if its diagonals are 17.8 and 30.7, and their angle is 80° 20′.

17. The area of a triangle is 4716 and two of its sides are 83 and 136. Find the angle between these sides.

18. The apothem of a regular decagon is 2. Find its area.

19. The sides of a parallelogram are 7.06 and 11.75, and the included angle is 66° 45′. Find its area.

20. The diagonals of a quadrilateral are 43.6 and 107.8 and their angle of intersection is 30° 55′. Find its area.

21. Each side of a rhombus is 7.5 and an angle is 73° 50′. Find its area.

CHAPTER VI

LOGARITHMS

PROPERTIES AND PRACTICE

87. The **logarithm** of a number is the index of the power to which another number, called the **base**, must be raised, to produce the given number.

In the equation $3^4 = 81$, the **base** is 3, the **number** is 81, and the logarithm is 4. Hence, the logarithm of 81, to the base 3, is 4, or, most briefly, $\log_3 81 = 4$.

Therefore, $B^l = N$ is the same statement as $\log_B N = l$ but in a different form.

The **common logarithm** of a number is the index of the power to which 10 must be raised to produce the given number.

In the equation $10^3 = 1000$, the base is 10, the number is 1000 and the logarithm is 3, or, most briefly, $\log_{10} 1000 = 3$.

88. Theorem. **The logarithm of the product of two quantities is equal to the sum of the logarithms of the quantities.**

Given: Two quantities M, N.

To Prove: $\log M \cdot N = \log M + \log N$.

Proof: $\left.\begin{array}{r}\text{Suppose } 10^x = M \\ \text{and } 10^y = N\end{array}\right\}; \therefore \left\{\begin{array}{ll}\log M = x & \text{(Def. of log).} \\ \log N = y & \text{(Def. of log).}\end{array}\right.$

Multiply, $10^{x+y} = M \cdot N$; $\therefore \log M \cdot N = x + y$ (Def. of log).

$\therefore \log M \cdot N = \log M + \log N$ (substitution).

89. Theorem. The logarithm of a fraction is equal to the logarithm of the numerator minus the logarithm of the denominator.

Given: The fraction $\frac{M}{N}$. **To Prove:** $\log \frac{M}{N} = \log M - \log N$.

Proof: Suppose $\left.\begin{array}{l} 10^x = M \\ 10^y = N \end{array}\right\}$; $\therefore \left\{\begin{array}{l} \log M = x \text{ (Why?).} \\ \log N = y \text{ (Why?).} \end{array}\right.$ (Suppose … and …)

Divide, $\quad 10^{x-y} = \frac{M}{N}$; $\therefore \log \frac{M}{N} = x - y$ (Why?).

$\therefore \log \frac{M}{N} = \log M - \log N$ (substitution).

90. Theorem. The logarithm of any power of a quantity is equal to the logarithm of the quantity multiplied by the exponent.

Given: The quantity M. **To Prove:** $\log M^p = p \cdot \log M$.

Proof: Suppose $10^x = M$; $\therefore \log M = x$ (Why?).

Raise each side to the pth power, $10^{px} = M^p$;

$\therefore \log M^p = px$ (?). $\therefore \log M^p = p \cdot \log M$ (substitution).

We may prove, similarly, that $\log \sqrt[r]{M} = \log M^{\frac{1}{r}} = \frac{1}{r} \log M$.

91. Theorem. The logarithm of unity is zero, and the logarithm of the base is unity.

Proof: I. $10^0 = 1$; $\therefore \log 1 = 0$ (Def. of log).

II. $10^1 = 10$; $\therefore \log 10 = 1$ (Def. of log).

Applications: Ex. 1. Write $\log \frac{5mn^3}{7x^2\sqrt{y}}$ as a polynomial.

$$\log \frac{5mn^3}{7x^2\sqrt{y}} = \log 5 + \log m + 3 \log n - \log 7 - 2 \log x - \tfrac{1}{2} \log y.$$

Ex. 2. Write $\log (a^2 - b^2)(x + y)^3$ as a polynomial.

$$\log (a^2 - b^2)(x + y)^3 = \log (a + b) + \log (a - b) + 3 \log (x + y).$$

Ex. 3. Reduce to a single term $3 \log c + \frac{1}{2} \log 5 - \log 4 - 2 \log x$.

$$3 \log c + \tfrac{1}{2} \log 5 - \log 4 - 2 \log x = \log \frac{c^3\sqrt{5}}{4x^2}$$

EXERCISE 51

Write each of the following in an expanded form:

1. $\log R^2 \times S^5$.
2. $\log x^2 \sqrt{y}$.
3. $\log 5xy$.
4. $\log \frac{7x^2}{3\sqrt{y}}$.
5. $\log \frac{ax^2}{11\sqrt{c}}$.
6. $\log 3a(x^2 - y^2)$.
7. $\log \frac{1}{2}\sqrt{2}$.
8. $\log \frac{5x\sqrt{7y}}{17a^3\sqrt[3]{c^2}}$.

Reduce each of the following to a single term:

9. $\log x + 3\log y$.
10. $3\log c + \log n - 2\log 5$.
11. $2\log p - \frac{1}{2}\log q$.
12. $3\log a - \log x - 5\log y$.
13. $2\log 2 + 3\log m - \frac{1}{2}\log 7 - 3\log x$.
14. $\log 3 + 2\log 5 - \frac{1}{2}\log 6 + \frac{1}{3}\log y$.

92. Parts of the logarithm of a number.

$\left.\begin{array}{l}10^0 = 1\\ 10^1 = 10\end{array}\right\}$ Hence the logarithm of any number between 1 and 10 is between 0 and 1.

$10^2 = 100$. Hence the logarithm of any number between 10 and 100 is between 1 and 2.

$10^3 = 1000$. Hence, etc.

$10^4 = 10{,}000$. Hence, etc.

In other words, the logarithm of a number of one figure is a fraction, usually expressed decimally. The logarithm of a number of two figures consists of the whole number "1" and a decimal. Thus $10^{1.30103} = 20$. $\therefore \log 20 = 1.30103$.

Likewise the logarithm of a number of 3 figures consists of the integer "2" and a decimal. Thus $10^{2.56229} = 365$. $\therefore \log 365 = 2.56229$, etc.

The integral part of a logarithm is called the **characteristic.**

The decimal part of a logarithm is called the **mantissa.**

It is obvious, then, **that the characteristic of the logarithm of a number is one less than the number of digits in the number at the left of the decimal point.**

For example; $\log 75 = 1.87506$; $\log 329 = 2.51720$; $\log 8.14 = 0.91062$, $\log 71.69 = 1.85546$.

93. Logarithms of decimal fractions.

$$10^0 = 1; \quad \therefore \log 1 = 0.$$

$$10^{-1} = .1; \quad \therefore \log .1 = -1 = 9.00000 - 10.$$

$$10^{-2} = .01; \quad \therefore \log .01 = -2 = 8.00000 - 10.$$

$$10^{-3} = .001; \quad \therefore \log .001 = -3 = 7.00000 - 10, \text{ etc.}$$

It is obvious, then, **that the characteristic of the logarithm of a decimal fraction is negative, and is (numerically) one greater than the number of zeros immediately following the decimal point.**

In practice, the logarithm of a decimal fraction is written as a binomial, the second term being -10, and the integral part of the first term is obtained by subtracting the number of zeros immediately following the decimal point from 9.

For example; $\log .46 = 9.66276 - 10$; $\log .00875 = 7.94201 - 10$.

NOTE. Instructions and directions as to the correct use of logarithmic tables are now usually given in connection with the tables. Such directions are omitted here, but exercises in practice are given, to familiarize the pupil with the handling of the tables. Examples in this book have been solved and the answers that are given have been obtained by use of five-place tables.

94. The mantissa of the logarithm of a number is independent of the position of the decimal point in that number. That is, **the mantissas of different numbers that have the same sequence of figures are the same, provided the base is 10.**

For example: $\log 5.6789$; $\log 56.789$; $\log 5678.9$ contain the same mantissas.

Proof. $$\log 5.6789 = \log \frac{56789}{10000} = \log 56789 - \log 10000$$

$$= 4.75426 - 4$$

$$= 0.75426.$$

$$\log 56.789 = \log \frac{56789}{1000} = \log 56789 - 3 = 1.75426.$$

$$\log 5678.9 = \log \frac{56789}{10} = \log 56789 - 1 = 3.75426.$$

Similarly it may be shown for any other numbers having the same sequence of figures, but different positions of the decimal point.

95. Antilogarithm. The number corresponding to a given logarithm is called the **antilogarithm**.

If $\log x = 2.52088$ and $\log 331.8 = 2.52088$, $x = 331.8$, and 331.8 is the number whose logarithm is 2.52088, or the antilog $2.52088 = 331.8$.

EXERCISE 52

In the following, tell which number is the base, which is the logarithm, which is the characteristic, which is *the number*.

1. $\log_7 49 = 2.$ **6.** $\log_{10} 75 = 1.88506.$

2. $\log_9 27 = 1.5.$ **7.** $m^{2.142} = x.$

3. $5^3 = 125.$ **8.** $a^{2.047} = 44.786.$

4. $4^{2.5} = 32.$ **9.** $5.021 = 71.5^{0.384}.$

5. $\log_{25} 125 = 1\frac{1}{2}.$ **10.** $4.7^{2.03} = 23.$

11. Express examples 3, 4, 7, 8, 9, 10 by use of the word "log."

12. Express examples 1, 2, 5, 6 as equations, not using the word "log."

Find from the tables the logarithms of the following numbers:

13. 580. **15.** 3590. **17.** .0386. **19.** .8279.

14. 5.8. **16.** 4.75. **18.** 4327. **20.** 746.2.

21. .00085.
22. 20050.
23. 3276.5.
24. 2.5628.
25. 6275.6.
26. 77.143.
27. .090764.
28. .80705.
29. 1.32856.
30. 8.95074.
31. 30.1145.
32. .0043077.

Find the numbers corresponding to the following logarithms, that is, find the antilogarithms.

33. 2.75815.
34. 0.99476.
35. 1.22427.
36. 4.41128.
37. 1.76695.
38. 0.36120.
39. 9.23452−10.
40. 8.82918−10.
41. 7.77379−10.
42. 8.92141−10.
43. 9.87911−10.
44. 6.60751−10.
45. 3.00245.
46. 8.63027−10.
47. 7.91300−10.
48. 2.30103.
49. 0.47725.
50. 1.77131.
51. 8.69951−10.
52. 2.05687.
53. 6.82018−10.
54. 0.42121.
55. 7.67018−10.
56. 9.95029−10.

57. Find the log 2.16629 and the antilog 2.16629.

58. Find the log 9.09228 and the antilog 9.09228 − 10.

96. Cologarithm. The **cologarithm** of a number **is** the logarithm of the reciprocal of that number.

$$\operatorname{colog} 256 = \log \tfrac{1}{256} = \log 1 - \log 256.$$

$$\begin{aligned} &\text{Now} & \log 1 &= 0.00000 = 10.00000 - 10, \\ &\text{and} & \log 256 &= \underline{2.40824} \\ & & \therefore \operatorname{colog} 256 &= 7.59176 - 10. \end{aligned}$$

The advantage of the use of the cologarithm is apparent upon illustration. The value of $\frac{49 \times 756}{83 \times 479} = 49 \times 756 \times \frac{1}{83} \times \frac{1}{479}$.

$$\begin{aligned} \text{The } \log \frac{49 \times 756}{83 \times 479} &= \log 49 + \log 756 - \log 83 - \log 479, \text{ or} \\ &= \log 49 + \log 756 + \operatorname{colog} 83 + \operatorname{colog} 479, \end{aligned}$$

and an example that combined addition and subtraction of logarithms is transformed into one employing only addition.

97. Computation by use of logarithms.

Ex. 1. Find x, if $x = \dfrac{538 \times 9.27}{496 \times .4075}$.

The pupil should form the habit of writing the scheme of the solution before looking up a single logarithm.

$$\begin{array}{rl}
\overbrace{\hphantom{\text{colog}\ .4075 =}}^{\text{Scheme}} & \overbrace{\hphantom{2.73078}}^{\text{Logarithms}} \\
\log 538 = & 2.73078 \\
\log 9.27 = & 0.96708 \\
\text{colog}\ 496 = & 7.30452 - 10 \\
\text{colog}\ .4075 = & \underline{0.38987\qquad\ \ } \\
\log x = & 11.39225 - 10 = 1.39225. \\
\therefore\ x = & 24.674^{+}.
\end{array}$$

Ex. 2. Find x, if $x = \sqrt[3]{73.146}$.

$$\begin{array}{r}
\overbrace{\hphantom{\log x = \tfrac{1}{3}\log 73.146 = 0.62139}}^{\text{Scheme}} \\
\log 73.146 = 1.86419, \\
\log x = \tfrac{1}{3}\log 73.146 = 0.62139\tfrac{2}{3} = 0.62140. \\
\therefore\ x = 4.1822^{+}.
\end{array}$$

Ex. 3. Find x, if $x = \sqrt[3]{.0073146}$.

$$\begin{array}{r}
\overbrace{\hphantom{\log x = \tfrac{1}{3}\log .0073146 = 9.28806}}^{\text{Scheme}} \\
\log .0073146 = 7.86419 - 10 = 27.86419 - 30, \\
\log x = \tfrac{1}{3}\log .0073146 = 9.28806 - 10. \\
\therefore\ x = .19411^{+}.
\end{array}$$

Ex. 4. Find x, if $x = 1.78\sqrt[4]{.9275}$.

$$\begin{array}{rl}
\overbrace{\hphantom{\tfrac{1}{4}\log .9275 =}}^{\text{Scheme}} & \\
\log 1.78 = & 0.25042 \\
\tfrac{1}{4}\log .9275 = & \underline{9.99183 - 10} \\
\log x = & .24225 \\
\therefore\ x = & 1.7468.
\end{array}$$

Ex. **5.** Find the value of $\sqrt[5]{\dfrac{9.7 \times .3256}{863.76 \times .0177}}$ by logarithms.

SCHEME

$$\begin{array}{rl}
\overbrace{\phantom{\text{colog }863.76}} & \\
\log 9.7 = & 0.98677 \\
\log .3256 = & 9.51268 - 10 \\
\text{colog } 863.76 = & 7.06361 - 10 \\
\text{colog } .0177 = & \underline{1.75203 \qquad\quad} \\
 & 19.31509 - 20 \\
 & \underline{30. \qquad\quad - 30} \\
 & 5\overline{)49.31509 - 50} \\
\log x = & 9.86302 - 10 \\
\therefore\ x = & .72948^{+}.
\end{array}$$

NOTE. A negative number can have no common logarithm. Hence, should negative quantities appear in an example to be computed by logarithms, the sign of the result is determined from the example, and not by the process of solution.

EXERCISE 53

Compute the value of each, by use of logarithms:

1. $\dfrac{97.3 \times 128.9}{436.5}$.

2. $\dfrac{71.48 \times 5.1473}{2.0716 \times 96.715}$.

3. $5.98^2 \times \sqrt[6]{76.14}$.

4. $\sqrt[3]{70.195} \times .8716$.

5. $-96.381 \div \sqrt{703.14}$.

6. $\sqrt{88.11 \times 9.706}$.

7. $\sqrt[3]{1.467 \times 38.192}$.

8. $\sqrt{876.34} \times \sqrt[3]{.0943}$.

9. $\sqrt{\dfrac{3.218 \times 9.7}{119.586}}$.

10. $\dfrac{-7\sqrt{528.3}}{\sqrt[3]{498.76}}$.

11. $\sqrt{\dfrac{7.63}{.527}} \times \sqrt[3]{\dfrac{85.44}{3.9174}}$.

12. $\sqrt{\dfrac{85.11 \times 7.637}{42.63 \times 1.976}}$.

13. $\sqrt[3]{\dfrac{9.177 \times .38825}{.76638 \times 753.11}}$.

14. $\dfrac{178.3\sqrt{9.1145}}{356.7\sqrt[3]{1498.7}}$.

15. $\sqrt[6]{\dfrac{38.76 \times 95.27}{76328 \times .07739^5}}$.

16. $\sqrt{\dfrac{1.234 \times 2.345 \times 3.456}{4.567 \times 5.678 \times .06789}}$.

17. $\left(\dfrac{-85.14 \times \sqrt[5]{.1479}}{7.1163 \times .095321}\right)^{\frac{2}{3}}$.

18. $\sqrt{.1} \times \sqrt[3]{.01} \times \sqrt[5]{.001}$.

19. $\sqrt{70} \times \sqrt[3]{700} \div \sqrt[5]{7000}$.

20. $(.003457)^{\frac{2}{7}} \div (.000903)^{\frac{3}{5}}$.

98. Problem. To derive a formula for finding the logarithm of a number to any base, having given the common tables of logarithms.

Derivation. Suppose $\log_B N = x$. $\therefore B^x = N$ (§ 87). Take the logarithm of each member, using the base 10,

$$\therefore x \log_{10} B = \log_{10} N. \quad \therefore x = \frac{\log_{10} N}{\log_{10} B}.$$

That is, $\log_B N = \dfrac{\log_{10} N}{\log_{10} B}$ (substitution).

Using the base a instead of 10, we obtain $\log_B N = \dfrac{\log_a N}{\log_a B}$.

99. Theorem. $\log_a B \times \log_B a = 1$.

Proof. $\log_B N = \dfrac{\log_a N}{\log_a B}$. (Established in § 98.)

Clearing of fractions, $\log_a B \times \log_B N = \log_a N$.

Substitute a for N, $\log_a B \times \log_B a = \log_a a = 1$. (See § 91).

100. An **exponential equation** is an equation in which the unknown quantity is an exponent.

Exponential equations of the form $a^x = b$ are readily solved by logarithms. Take the logarithm of each member of the equation, $x \log a = \log b$. $\therefore x = \dfrac{\log b}{\log a}$.

Note. There should be no confusion between $\log \frac{a}{b}$ and $\frac{\log a}{\log b}$. The former $= \log a + \text{colog}\, b$; the latter $= \log a \div \log b$.

Ex. 1. Solve $17^x = 150$ for x.

Taking the logarithm of each member, $x \log 17 = \log 150$.

$$\therefore x = \frac{\log 150}{\log 17} = \frac{2.17609}{1.23045} = 2.17609 \div 1.23045 = 1.768^+.$$

Ex. 2. Find $\log_{12} 550$.

Let $\log_{12} 550 = x$. $\therefore 12^x = 550$. Hence, $x \log 12 = \log 550$.

$$\therefore x = \frac{\log 550}{\log 12} = \frac{2.74036}{1.07918} = 2.53^+.$$

That means, that $12^{2.53} = 550$, approximately.

Ex. 3. Find $\log_5 .3$.

Let $\log_5 .3 = x$. $\therefore 5^x = .3$. Hence $x = \dfrac{\log .3}{\log 5} = \dfrac{9.47712 - 10}{.69897}$

$= \dfrac{-.52288}{.69897} = -.748$. That means, that $5^{-.748} = .3$, approximately.

EXERCISE 54

Solve the following exponential equations for x:

1. $19^x = 527$.
2. $2.8^x = 200$.
3. $92.4^x = 187.5$.
4. $87.41^x = 97.88$.
5. $11^{x+1} = 375.4$.
6. $18.3^{2x} = 975.5$.
7. $42^{2x-1} = 576.84$.
8. $9.3^x = 5.7$.
9. $12^x = 3.7$.
10. $2.8^x = .095$.
11. $.0314^x = 8.34$.
12. $.6956^x = .8432$.

Find, by using an exponential equation, the value of each:—

13. $\log_7 39$.
14. $\log_{30} 763$.
15. $\log_4 30$.
16. $\log_6 110$.
17. $\log_{12} 500$.
18. $\log_{2.5} 8.5$.
19. $\log_{3.4} .951$.
20. $\log_5 .084$.
21. $\log_{.8} 6.2$.
22. $\log_{50} 2.5$.
23. $\log_{25} 100$.
24. $\log_{\frac{1}{2}} \frac{1}{10}$.

101. Application to compound interest. If A = amount, P = principal, r = rate, n = number of years, the amount of a certain principal at compound interest in a given time is expressed by the formula: $A = P\,(1 + r)^n$. Each member of this equation is adapted to logarithmic computation. There may be four types of examples dealing with this formula, according as the unknown quantity is A, or P, or r, or n. If n is the unknown quantity, and A, P, and r are known, the formula becomes an exponential equation.

Ex. 1. What principal will amount to $10,000 in 5 years, at 6% compound interest?

SOLUTION: The formula becomes: $P = \frac{A}{(1+r)^n} = \frac{10000}{(1.06)^5}$.

$$\begin{aligned} \log 10000 &= 4.00000 \\ 5 \operatorname{colog} 1.06 &= \underline{9.87345 - 10} \\ \log P &= 3.87345. \\ \therefore P &= \$7472.20. \end{aligned}$$

Ex. 2. In how many years will $5000 amount to $8000 at $4\frac{1}{2}$% compound interest?

SOLUTION: The formula becomes: $(1+r)^n = \frac{A}{P}$.

That is $(1.045)^n = \frac{8000}{5000} = 1.6$.

$$\therefore n = \frac{\log 1.6}{\log 1.045} = \frac{.20412}{.01912} = 10.675 \text{ yr.}$$

EXERCISE 55

1. Find the amount of $5000 in 8 yr. at compound interest, the rate being 5%.

2. Find the amount of $6750 in 10 yr. at 4% compound interest.

3. Find the principal that will amount to $10,000 in 6 yr. at 3% compound interest.

4. Find the principal that will amount to $5000 in 15 yr., compound interest at 5%.

5. In what time will $2000 amount to $5000 at 6% compound interest?

6. How long must $8000 be at compound interest to amount to $12,000, the rate being $5\frac{1}{2}$%?

7. What is the rate when $500 amounts to $1500 in 20 yr., compound interest?

8. At what rate must I invest $8560 at compound interest that in 16 yr. I may withdraw $15,000?

9. In how many years will a given sum of money double itself at compound interest, the rate being 3 %? 4 %? 5 %? 6 %? 7 %?

102. Solution of problems involving exact powers of numbers or fractions is often easily effected without the use of the tables.

Ex. 1. If $\log 4 = .60206$, find $\log 16$, $\log 2$, $\log .8$, $\log 5000$.

Solution: (*a*) $\log 16 = \log 4^2 = 2 \log 4 = 1.20412.$

(*b*) $\log 2 = \log 4^{\frac{1}{2}} = \frac{1}{2} \log 4 = .30103.$

(*c*) $\log .8 = \log 8 + \text{colog } 10 = 3 \log 2 + 9.00000 - 10$

$= 9.90309 - 10.$

Or, $\log 8 = 3 \log 2 = 0.90309$. The log .8 would require $9.00000 - 10$ for its characteristic; $\therefore \log .8 = 9.90309 - 10.$

(*d*) $\log 5 = \log \frac{10}{2} = \log 10 + \text{colog } 2 = 0.69897.$

$\therefore \log 5000 = 3.69897.$

Ex. 2. If $\log 3 = .47712$ and $\log 7 = .84510$, find $\log 210$, $\log \sqrt{2.1}$, $\log (2\frac{1}{3})^3$, $\log \sqrt[3]{6.3}$.

Solution: (*a*) $\log 21 = \log 3 + \log 7 = 1.32222.$

$\therefore \log 210 = 2.32222.$

(*b*) $\log 2.1 = 0.32222.$

$\therefore \log \sqrt{2.1} = \frac{1}{2} \log 2.1 = 0.16111.$

(*c*) $\log 2\frac{1}{3} = \log \frac{7}{3} = \log 7 + \text{colog } 3 = .36798.$

$\therefore \log (2\frac{1}{3})^3 = 3 \log 2\frac{1}{3} = 1.10394.$

(*d*) $\log 63 = \log 3^2 + \log 7 = 1.79934.$

$\therefore \log \sqrt[3]{6.3} = \frac{1}{3} \log 6.3 = \frac{1}{3}(.79934) = .26645.$

Ex. **3.** Solve $27^x = 243$; and find $\log_{64} 16$.

SOLUTION: (*a*) $27^x = 243$ may be written $(3^3)^x = 3^5$ or $3^{3x} = 3^5$.

$$\therefore 3x = 5 \text{ and } x = 1\tfrac{2}{3}.$$

(*b*) Let $\log_{64} 16 = x$. $\therefore 64^x = 16$ or $2^{6x} = 2^4$. $\therefore x = \frac{2}{3}$.

$$\text{That is, } \log_{64} 16 = \tfrac{2}{3}.$$

EXERCISE 56

This entire exercise is to be solved without the use of the logarithmic tables.

1. Distinguish between log 3 and log 300. Between log 4 and log 40. Between log 6 and log .006. Between log 6.7 and log 6700. Between log 5 and log 5^3. Between log 6 and log 360 Between log 2 and log $\frac{1}{5}$.

2. If $\log 13 = 1.11394$, write log 130; log .0013. Write the number whose logarithm is 0.11394.

3. If $\log 874.6 = 2.94181$, write log 87460 and antilog 0.94181.

4. If $\log 3246 = 3.51135$ and $\log 3.247 = 0.51148$, find log 32.465 and antilog $7.51143 - 10$.

5. If $\log 50.27 = 1.70131$ and $\log 502.8 = 2.70140$, find log .050278 and antilog 3.70136.

Given $\log 2 = 0.30103$, find the logarithms of the following numbers:

6. 4; 40; 400; .04; $\sqrt[3]{4}$; $\sqrt[4]{.004}$; $\sqrt{4000}$.

7. 8; 80; $\sqrt{8}$; $\sqrt[3]{80}$; $\sqrt{.08}$; 16; 160; $\sqrt[3]{16}$; $\sqrt{1.6}$.

8. 32; 6.4; 12.8; 25.6; 3200; $\sqrt[3]{6400}$; $\sqrt{1.28}$; $\sqrt{.032}$.

9. 5; 50; 2.5; $\frac{1}{2}$; $12\frac{1}{2}$; $\sqrt{500}$; $\frac{1}{4}$; $\frac{5}{8}$; $\frac{25}{32}$; $1\frac{3}{5}$.

Given $\log 2 = 0.30103$ and $\log 3 = 0.47712$, find:

10. log 6; log 9; log 2.7; log 12; log 1.5; log 162.

11. log 60; log 150; log 2.4; log 45; log 14.4.

12. log $\frac{1}{3}$; log $\frac{1}{6}$; log $\frac{5}{6}$; log $\frac{2}{3}$; log $\frac{5}{9}$; log $\frac{24}{25}$.

Having given log 2, log 3, log 7, explain how to determine the logarithm of each of the following:

13. 4, 5, 6, 8, 9, 15, 16, 18, 21, 25, 28, 35, 36, 40.

14. 42; 45; 210; .042; 6.3; .028; 14×3.5; 60×49.

15. $.09 \div \sqrt[3]{1.89}$; $490 \div 105$; $\sqrt{\frac{2}{3}} \times \sqrt[3]{\frac{7}{6}}$.

Find the unknown quantity in each:

16. $8^x = 64$.

17. $8^x = 128$.

18. $x^{-\frac{4}{3}} = 16$.

19. $16^x = 8$.

20. $25 = 125^x$.

21. $27 = 81^{-x}$.

22. $64 = 32^x$.

23. $\log_5 25$.

24. $\log_{27} 81$.

25. $\log_{\frac{1}{9}} 27$.

26. $\log_{216} 36$.

27. $\log_x 9 = \frac{2}{3}$.

28. $\log_x \frac{1}{27} = -\frac{3}{5}$.

29. $\log_8 x = 1\frac{2}{3}$.

30. $\log_{\frac{4}{9}} x = -2.5$.

NATURAL FUNCTIONS

EXERCISE 57

Find, from the table of Natural Functions, the actual values of

1. sin 17° 40′.

2. cos 22° 50′.

3. tan 70° 10′

4. sin 81° 18′.

5. cot 38° 17′.

6. tan 28° 25′ 30″.

7. sin 65° 30′ 45″.

8. cos 57° 7′ 20″.

9. sin 40° 41′ 36″.

10. cot 85° 19′ 20″.

Find the angle if:

11. $\sin A = .27004$.

12. $\cos A = .85911$.

13. $\tan A = 1.4478$.

14. $\cot A = 6.3496$.

15. $\sin A = .99133$.

16. $\sin A = .21175$.

17. $\cos A = .78436$.

18. $\tan A = 1.6289$.

19. $\cot A = .35600$.

20. $\cos A = .44812$.

21. Explain how to find sec 25° 15′.

22. Explain how to find csc 71° 20′.

23. If $\sec A = 1.3456$, how could A be found?

24. If $\csc A = 2.1178$, how could A be found?

LOGARITHMS OF THE TRIGONOMETRIC FUNCTIONS

EXERCISE 58

Find, from the tables, the logarithms of:

1. $\sin 32^\circ 17'$.

2. $\tan 41^\circ 38'$.

3. $\cos 22^\circ 48'$.

4. $\sin 56^\circ 40'$.

5. $\cot 40^\circ 10'$.

6. $\cos 70^\circ 15'$.

7. $\sin 15^\circ 16' 30''$

8. $\sin 38^\circ 41' 40''$

9. $\tan 40^\circ 17' 24''$

10. $\tan 25^\circ 25' 25''$.

11. $\sin 55^\circ 45' 30''$.

12. $\sin 73^\circ 28' 50''$

13. $\tan 80^\circ 5' 36'$.

14. $\cos 15^\circ 16' 30''$.

15. $\cos 35^\circ 27' 15''$.

16. $\cos 68^\circ 18' 24''$.

17. $\cot 50^\circ 30' 48''$.

18. $\cot 61^\circ 16' 10''$.

Also find:

19. colog $\sin 33^\circ 21' 32''$.

20. colog $\cos 27^\circ 19' 44''$.

21. colog $\tan 40^\circ 6' 16''$.

22. colog $\cot 15^\circ 25' 45''$.

23. colog $\tan 77^\circ 41' 18''$.

24. colog $\cot 63^\circ 23' 25''$.

25. colog $\sin 53^\circ 17' 50''$.

26. colog $\cos 80^\circ 29' 36''$.

27. Explain how to find $\log \sec 10^\circ 20' 30''$.

28. Explain how to find $\log \csc 40^\circ 27' 15''$.

29. The logarithm of what function was really found in Ex. 19? in Ex. 20? in Ex. 21? in Ex. 22?

30. If $\log \cos 77^\circ 28' = 9.33647 - 10$, and $\log \cos 77^\circ 29'' = 9.33591 - 10$, find $\log \cos 77^\circ 28' 45''$

Find the angle A, if:

31. $\log \sin A = 9.48014 - 10$.

32. $\log \cos A = 9.96201 - 10$.

33. $\log \tan A = 9.82762 - 10$.

34. $\log \cot A = 0.04836$.

35. $\log \sin A = 9.88896 - 10$.

36. $\log \tan A = 0.13954$.

37. $\log \cos A = 9.65878 - 10$.

38. $\log \sin A = 9.60322 - 10$.

39. $\log \sin A = 9.67831 - 10$.

40. $\log \tan A = 9.67730 - 10$.

41. $\log \tan A = 9.70553 - 10$.

42. $\log \sin A = 9.98367 - 10$.

43. $\log \tan A = 0.41347$.

44. $\log \cos A = 9.96586 - 10$.

45. $\log \cos A = 9.92087 - 10$.

46. $\log \cos A = 9.89296 - 10$.

47. $\log \cot A = 0.44325$.

48. $\log \cot A = 0.11151$

49. $\log \cos A = 9.59538 - 10$

50. $\log \cot A = 9.20428 - 10$.

51. $\log \tan A = 0.45353$.

52. $\log \sin A = 9.98710 - 10$.

53. Having the log sec A given, how may we find A?

54. If $\log \sec A = 0.13753$, find A.

55. If $\log \csc A = 0.33876$, find A.

56. If $\log \sec A = 0.38811$, find A. Find $\log \cot A$.

57. If $\log \tan A = 0.58900$, find $\log \sin A$ and $\log \cos A$.

Find the angle θ from the following equations:

58. $\cos\theta = \sqrt{97.38 \times .00568}$. **59.** $\sin\theta = \dfrac{27.5 \times 3.4627}{9.1176 \times 53.07}$.

60. $\tan\theta = \sqrt{1.078 \cot 76^\circ\, 14'\, 18''}$.

61. $\sin\theta = \sqrt[3]{.7638 \tan 38^\circ\, 15'\, 47''}$.

62. $\cos\theta = \sqrt{\tan 43^\circ\, 17'\, 50'' \times \sin 63^\circ\, 25''}$.

63. If $\log \tan A = 0.21796$, $\log \tan 58^\circ\, 48' = 0.21780$ and $\log \tan 58^\circ\, 49' = 0.21808$, find A.

GENERAL REVIEW

EXERCISE 59

1. Find the logarithm of 77 to the base 7, to two places of decimals.

2. Find, without tables, the logarithm of 128 to the base 32.

3. If $\sec x = -2$ and $\tan x$ is positive, find $\sin x$ and $\cos 2x$.

4. Find the six functions of 150° and of −150°.

5. Prove: $\sqrt{1 + \sin 2x} - \sqrt{1 - \sin 2x} = 2 \sin x$. (For the "1" substitute $\sin^2 x + \cos^2 x$.)

6. How many degrees are there in the angle at the center of a circle if the chord of the intercepted arc is $\frac{3}{4}$ of the radius?

7. If $\sin \theta = \frac{1}{5}$ and $\cos \phi = \frac{1}{3}$, find $\cos(\theta + 2\phi)$.

8. In a triangle whose sides are 7.056, 8.417, 9.763, find the greatest angle and the least angle.

9. Explain fully how to find the lengths of the four diagonals that can be drawn from any vertex of a regular 7-gon, each side of which is 12.

10. Change to degrees: $\frac{3}{2}\pi$ radians; $\frac{7}{12}\pi$ radians; $\frac{7}{8}\pi$ radians; $\frac{\pi - 1}{3}$ radians.

11. Prove the identity: $\frac{\cot A \cos A}{\cot A + \cos A} = \frac{\cot A - \cos A}{\cot A \cos A}$.

$\left(\text{Reduce each member separately, to its simplest form, } \frac{1 - \sin A}{\cos A}.\right)$

12. Prove the formula: $\sin R + \sin S = 2 \sin \frac{1}{2}(R + S) \cos \frac{1}{2}(R - S)$ and the other three resembling this. State these four formulas in words.

13. If $x = \pm\sqrt{\frac{1}{2}}$, find all values less than 2π, for $\sin^{-1}x + \cos^{-1}x$.

14. If $\tan A = \sin^2 80° \, 16' \times \cos^2 15° \, 45'$, find A.

15. If $\tan^2(\pi - \theta) - \sec(\pi + \theta) = 5$, find $\cos\theta$.

16. Find the values of: $\cos(\frac{1}{2}\sin^{-1}\frac{1}{2}\sqrt{2})$.

17. Is $2\sin x - \cos x = 3$ a possible equation?

18. If $\cot A = \sqrt{\cos 39° \, 35' \, 56''}$, find A and $\log\sin 2A$.

19. Prove the identity:

$$\frac{1+\sin x-\cos x}{1+\sin x+\cos x}+\frac{1+\sin x+\cos x}{1+\sin x-\cos x}=2\csc x.$$

20. A rectangle is 17×33 in. A line is drawn joining the midpoints of two adjacent sides. Find its length and the acute angles at its ends.

21. How far must a person be elevated above the earth's surface that he may see an object on the earth's surface, and at a distance of 120 mi. from him? (Radius of earth = 3960 mi.)

22. Solve for x, $2\cos^2 x + 11\sin x - 7 = 0$.

23. Prove that: $\dfrac{\sin 215° - \sin 55°}{\cos 215° - \cos 55°} = 1$.

24. Prove the formulas employed in solving a triangle if two of the sides and their included angle are known. State these formulas in words.

25. Given $\sin\theta = \frac{2}{3}$, find all six functions of 2θ and of $\frac{1}{2}\theta$, (*a*) if θ is in the first quadrant; (*b*) if θ is in the second quadrant.

26. Find the value of $\sin(\sin^{-1}\frac{1}{2} + \sin^{-1}\frac{1}{4})$.

27. Change 72° to radians. Also 138°; and −207°.

28. From the top of a lighthouse 208 ft. above sea level the angles of depression of two boats in a vertical plane with the lighthouse are observed to be 62° and 26°. What is the distance between the boats?

29. Explain how to find the lengths of all of the diagonals of a regular decagon, having given the length of one side.

30. Prove $\cos^2 x + \sin^2 x \sin^2 y + \sin^2 x \cos^2 y = 1$.

31. Compute by logarithms the value of $\dfrac{6.458 \cos 48^\circ\, 27'\, 50''}{9.7032 \sin 21^\circ\, 14'\, 45''}$.

32. If $\log 20 = 1.30103$, find, without tables, $\log 40$, $\log 50$, $\log .8$, $\log \frac{2}{5}$, and $\log 1\frac{1}{4}$.

33. Find the value of $\tan(2 \sec^{-1} \sqrt{3}) - \sec\left(\frac{1}{2} \sin^{-1} \dfrac{1}{\sqrt{3}}\right)$.

34. Prove: $\dfrac{\sqrt{1 + \sin 2A} - \cos A}{\sqrt{1 + \sin 2A} - \sin A} = \tan A$.

35. If $2 \sin^2 \alpha = 3 \cos \alpha$, find α.

36. What is the angle of the sun's elevation if a man is $\frac{3}{4}$ the length of his shadow?

37. The diagonals of a parallelogram are 136 and 255 feet long and they intersect at an angle of 31° 18′. Find the sides, the angles, and the area of this parallelogram.

38. The latitude of Philadelphia is 40°. If the earth's radius is 3960 mi., what is the radius of the parallel of latitude through Philadelphia?

39. Prove: $2 \sin^2(\frac{1}{4}\pi - \frac{1}{2}x) = 1 - \sin x$. (Consult § 67.)

40. Find θ if $\sin \frac{1}{2}\theta = \cos 4\theta$.

41. If the earth's radius is given as 3960 mi., what is the difference of the latitude of two cities, one a thousand miles north of the other?

42. Solve for θ: $\sin\theta + \sin 2\theta + \sin 3\theta = 1 + 2\cos\theta$.

43. Prove: $\tan\left(\dfrac{\pi}{6} + m\right) \tan\left(\dfrac{\pi}{6} - m\right) = \dfrac{2\cos 2m - 1}{2 \cos 2m + 1}$.

44. If an arc is 8.1 ft. long, and the radius is 6 ft., find the central angle in radians; in degrees.

45. Prove: $2 \cos^2(\frac{1}{4}\pi - \frac{1}{2}x) = 1 + \sin x$.

46. How can one find log csc 48° 20′ from the tables? log csc 110° 50′? log sec 72° 28′?

47. Prove: $\cot b \pm \cot a = \frac{\sin(a \pm b)}{\sin a \sin b}$.

48. Show that $\frac{\sin 25° + \sin 15°}{\cos 25° - \cos 15°} = -\cot 5°$.

49. Upon a tower 212 ft. high stands a pole which at a horizontal distance of 150 ft. from the base of the tower subtends an angle of 7°. What is the length of the pole?

50. From one end of a diameter of a circle a chord $\frac{2}{3}$ the length of the diameter is drawn. Find the angle between these two lines.

If C is a right angle in △ ABC, prove these ten relations:

51. $\sin 2A = \sin 2B$.

52. $a^2 \tan B = b^2 \tan A$.

53. $\tan \frac{1}{2} A = \frac{c-b}{a}$.

54. $\cos 2A = \frac{b^2 - a^2}{c^2}$.

55. $\cos(B - A) = \frac{2ab}{c^2}$.

56. $\sin(B - A) = \cos 2A$.

57. $\tan(45° + A) = \frac{b+a}{b-a}$.

58. $\sin^2 \frac{1}{2} A = \frac{c-b}{2c}$.

59. $\sin^2 A + \sin^2 B = 1$.

60. $b^3 \sin A + a^3 \sin B = a \cdot b \cdot c$.

61. Find the length of the diagonal of a regular pentagon each side of which is 20.

62. Prove that $\tan(45° - \frac{1}{2}x) = \sqrt{\frac{1 - \sin x}{1 + \sin x}}$. (Consult § 67.)

63. Find the values of: $\sin 210°$; $\cos \frac{2\pi}{3}$; $\tan \frac{4\pi}{3}$; $\sec 300°$.

64. Prove: $\frac{\cos(x - 3y) - \cos(3x - y)}{\sin 2x + \sin 2y} = 2 \sin(x - y)$.

65. Solve for θ, $\tan(45° - \theta) + \tan(45° + \theta) = 4$.

66. Explain how to find the functions of: 15°, 75°, 22° 30′, 105°, 120°, −30°.

67. Prove that $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$, if it is given that $A + B + C = 180°$, that is, if A, B, C are the angles of a triangle.

68. What angle less than 360° has the same tangent as 219°? sine as 200°? cotangent as 130°? cosine as 100°? secant as 27°? sine as 85°?

69. Solve the triangle ABC, if $A = 38° \; 28' \; 12''$, $a = 4.711$, $c = 6.078$.

70. If $\cos^2 \frac{2}{3} A = \frac{1}{4}$, find all values of $A < 2\pi$.

71. If $\sqrt[4]{\cos A} = \dfrac{98.713 \times 467.38}{538.16 \times 87.045}$, find A.

72. If $\tan \alpha = \dfrac{b}{a}$, find the value of $a \cos 2\alpha + b \sin 2\alpha$.

73. Find one value of x for each:

$\sin x = \cos 20°$. $\sin x = \cos(-95°)$. $\cos 3x = \sin 54°$.

$\cot x = \tan \dfrac{2\pi}{5}$. $\tan 2x = \cot 100°$. $\sin \frac{3}{2} x = \cos \dfrac{7\pi}{6}$.

74. What is the latitude of that city whose parallel of latitude has a radius equal to half the radius of the earth?

75. In a circle whose radius is 50 is inscribed a regular polygon of 75 sides. Find the length of a side, the apothem, and the area of this polygon.

76. The length of a lake from a certain viewpoint subtends an angle of 38° 46′, and the distances from this point to the ends of the lake are 470 yd. and 345 yd., respectively. What is the length of the lake?

77. Prove: $\tan 50° + \cot 50° = 2 \sec 10°$.

78. If the sides of a triangle are 328, 417, 509, find the largest angle and the area. Verify results by a formula for area.

79. Given: $\log 14 = 1.14613$, $\log 15 = 1.17609$, $\log 16 = 1.20412$; find the logarithms of the numbers from 1 to 10.

80. In a triangle ABC, $a = 9.716$, $A = 47° \; 18' \; 25''$, $C = 53° \; 39' \; 32''$. Find b and c.

81. Determine the number of solutions in each △:

(*a*)	(*b*)	(*c*)
$a = 13.4$,	$c = .58$,	$b = 109$,
$b = 11.46$,	$a = .75$,	$a = 94$,
$A = 77° \; 20'$	$C = 60°$.	$A = 92° \; 10'$.

82. Reduce the law of sines and law of cosines in the case of an isosceles triangle.

83. Solve the triangle: $a = .4394$, $b = .9763$, $c = .8535$.

84. Solve the triangle: $c=5.114$, $A = 63° 71' 46''$, $B=78° 14' 26''$

85. Prove, geometrically:

$$\cos(90° + x) = -\sin x. \qquad \sin(-x) = -\sin x.$$
$$\tan(180° - x) = -\tan x. \qquad \cos(270 + x) = \sin x.$$

86. Evaluate: $\dfrac{\sin(90+\theta) - \cos(-\theta) + \tan 180° + \cos^2 330°}{\tan 315° + \tan^2 330°}$.

87. Prove: $\cos x(2\sec x + \tan x)(\sec x - 2\tan x) = 2\cos x - 3\tan x$.

88. Prove: $2\sin A\cos B = \sin(A+B) + \sin(A-B)$, and write like expressions for $2\cos A\sin B$, $2\cos A\cos B$, and $2\sin A\sin B$.

89. What is the base of the system of logarithms in which $\log 3 = .24097$?

90. If $\cos 2\theta = 2\sin\theta$, find the value of $\sin\theta$ and $\cos\frac{1}{2}\theta$.

91. Express $\tan A$ in terms of $\sin A$; in terms of $\cos A$; of $\cot A$.

92. Express $\cos A$ in terms of each of the other functions.

93. The sides of a triangle are 83.7, 91.6, 75.4. Find its area and its largest angle.

94. Mention two advantages, one referring to the characteristic, the other to the mantissa, obtained by using 10 as the base of a system of logarithms.

95. If $\log 2 = .30103$, $\log 3 = .47712$, find $\log 180$, $\log \frac{1}{6}$, $\log\sin 30°$, $\log\tan 30°$, $\log\cos 30°$, $\log\sin 45°$.

96. Prove: $\sec 2x + \tan 2x + 1 = \dfrac{2}{1 - \tan x}$.

97. Show that: $\cos 3x + \cos 5x + \cos 7x + \cos 15x$
$= 4\cos 4x\cos 5x\cos 6x$.

98. Prove: $\tan A(\cot\frac{1}{2}A - \tan\frac{1}{2}A) = 2$. (Consult § 67.)

99. Construct the angles whose sine is $\frac{2}{5}$. The angles whose tangent is $1\frac{2}{3}$. The angles whose cosine is $-\frac{3}{7}$.

100. Three circles whose radii are 11, 13, 18 touch one another externally. Find the angles between the lines that join their centers.

101. A right triangle has angle $A = 30°$, $C = 10\sqrt{3}$. The right angle C is bisected by CD, meeting the hypotenuse at D. DE is drawn perpendicular to AC. Find the length of DE without the use of the tables.

102. Prove: $(\tan a \cot 2a + 1) \div (\cot 2a + \cot a) = \tan a$.

103. Prove: $\dfrac{\cos m - \sin m}{\cos m + \sin m} = \sec 2m - \tan 2m$.

104. Solve: $3 \sec^2 \theta - 2\sqrt{3} \tan \theta = 6$ for values of $\theta < 360°$

105. Prove: $\sin x (\cot \frac{1}{2} x + \tan \frac{1}{2} x) = 2$.

106. From the top of a building 150 ft. high the angles of depression of the top and base of a monument standing upon the same level as the building are 30° and 45° respectively. Find the height of the monument.

107. Prove that, in any triangle, $\tan A = \dfrac{a \sin C}{b - a \cos C}$.

108. The angles of elevation of the top of a tower on a horizontal plane, observed from two points in the plane, a ft. and b ft. respectively from the foot of the tower and in the same line with it, are found to be complementary. Show that the height of the tower is $\sqrt{a \cdot b}$ ft.

109. Prove: $\cos^2 (a + x) - \sin^2 a = \cos x \cdot \cos (2a + x)$.

110. Find log 700 to the base 70.

111. What is the length of an arc of a circle whose radius is 8.4, intercepted by a central angle of 2.4 radians?

112. If $\tan x = \dfrac{2ab}{a^2 - b^2}$, find $\sin x$, $\sin \frac{1}{2} x$, and $\tan 2x$.

113. Given $\log \cot A = 0.32307$ and A an angle in the third quadrant. Find A and find $\operatorname{colog} \cos \frac{1}{2} A$.

114. Solve: $\sec^2 y \csc^2 y + 2 \csc^2 y = 8$.

115. Find the value of x, by logarithms, if $(1.036)^{3x} = 1.235$.

116. Find x, by logarithms, if $3 \cot x = \sqrt{0.5}$.

117. Write as a polynomial: $\log \frac{1}{2}\sqrt{2}$; $\log \pi R^2$; $\log \frac{1}{3}a^3 \sqrt[3]{x}$.

118. Compute θ, if $4^{\cot \theta} = 3$.

119. The vertex angle of an isosceles triangle is 111° 26′ 58″, and the equal sides are each 415.18. Compute the length of the base and the altitude upon it.

120. Given, $\log_9 10 = 1.0480$. Find, without tables, $\log_9 30$; $\log_9 270$; $\log_{10} 81$.

121. Each side of a regular pentagon is 8 ft. Find the length of a diagonal.

122. Solve: $8 x^5 = 3.1516$, for x.

123. Solve: $60 \sin^2 \alpha = \sqrt[3]{.87153}$, for α.

124. Solve: $5 \cos^2 2\theta = \sqrt{\tan 75° \, 40'}$, for θ.

125. Given the three dimensions of a rectangular room. Explain how one can find the plane angles made at one of the vertices by a solid diagonal of the room.

[A solid diagonal is a line joining two vertices not in the same bounding plane.]

126. If $\sin 20° = .342$, how can one find $\sin 40°$? $\sin 10°$? $\cos 40°$? $\tan 10°$? $\sin 5°$? $\cos 25°$?

127. Prove: $1 + \tan 2x - \sec 2x = \dfrac{2}{1 + \cot x}$.

128. What is the area of a triangle two of whose sides are 8.008 and 12.93, and their included angle, 110° 38′ 40″?

129. One side of a regular n-gon is a. Prove that its area $= \frac{1}{4} n \cdot a^2 \cdot \cot \dfrac{180°}{n}$.

130. Solve the equation, $\cot x \tan 2x = 3$.

131. Solve: $\sin 4x - \cos 3x = \sin 2x$.

132. If $x \sin \theta = \sqrt{3}$ and $x \cos \theta = 3$, find θ and x.

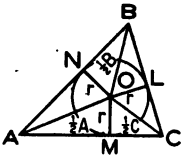
133. If $\tan 2x = \frac{12}{5}$, find $\sin \frac{1}{2}x$ and $\tan \frac{1}{2}x$.

134. Solve $\sin 2x \cot x = .5 + \sin^2 x$.

135. Prove that the area of a right triangle whose hypotenuse is $c = \frac{1}{4}c^2 \sin 2A$.

136. Prove that the radius of a circle inscribed in a triangle equals $\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$.

137. Prove, in the adjoining figure, that $AM = AN = s - a$; $BL = BN = s - b$; $CL = CM = s - c$.



138. Derive the formulas of § 79 geometrically, using the adjoining figure.

139. Prove: $\cos^2(a - b) - \sin^2(a + b) = \cos 2a \cdot \cos 2b$.

140. Prove: $\dfrac{\tan x + \sec x + 1}{\tan x + \sec x - 1} = \dfrac{\sec x + 1}{\tan x}$.

141. Show by a figure, that $\cos(180° + A) = \cos(180° - A)$, and $\tan(90° + A) = -\cot(180° + A)$.

142. Derive a formula for $\sin 5x$ in terms of $\sin x$.

143. Derive a formula for $\cos 5x$ in terms of $\cos x$.

144. The sides of a triangle are 5, 6, 9. Find the greatest angle and the least angle, if:

$$\log 2 = .30103, \qquad \log \tan 15° \, 47' = 9.45126 - 10,$$
$$\log 3 = .47712, \qquad \log \tan 15° \, 48' = 9.45174 - 10,$$
$$\log \tan 54° \, 44' = 0.15048,$$
$$\log \tan 54° \, 45' = 0.15075.$$

145. If x is in quadrant III, and $\csc x = -\sqrt{5}$, show that $\dfrac{\tan x - \cot x}{\tan x + \cot x} = -\frac{3}{5}$.

146. Prove: $\csc^{-1}\sqrt{5} + \tan^{-1}\frac{1}{3} = 45°$.

147. Prove: $\tan x - \tan \frac{1}{2}x = \tan \frac{1}{2}x \sec x$.

148. $\dfrac{\cos(\theta + 45)}{\cos(\theta - 45)} = \sec 2\theta - \tan 2\theta$.

149. Prove: $\cot\beta - \cot 2\beta = \csc 2\beta$.

150. Which is the greater, 126° or 2.3 radians?

151. Prove: $\tan\frac{1}{2}n + 2\sin^2\frac{1}{2}n\cdot\cot n = \sin n$.

152. Prove: $\dfrac{\tan 2x + \tan x}{\tan 2x - \tan x} = \dfrac{\sin 3x}{\sin x}$.

153. Show that $\sqrt{1+\sin\alpha} + \sqrt{1-\sin\alpha} = 2\sin\frac{1}{2}\alpha$;

and $$\sqrt{1+\sin\alpha} - \sqrt{1-\sin\alpha} = 2\cos\frac{1}{2}\alpha.$$

154. Distinguish between a trigonometric equation and a trigonometric identity. Of the following prove the identity and solve the equation:

(*a*) $\sin^2 x + \cos x = 1\frac{1}{4}$.

(*b*) $1 + \cos x - \sin x = \sqrt{2(1+\cos x)(1-\sin x)}$.

155. Show that $\sin 7\frac{1}{2}^\circ + \sin 67\frac{1}{2}^\circ + \sin 127\frac{1}{2}^\circ + \sin 187\frac{1}{2}^\circ = \sqrt{3}\cdot\cos 7\frac{1}{2}^\circ$.

156. Is this relation true for all values of m?
$$\tan^2 m - \sin^2 m = \tan^2 m \sin^2 m.$$

157. Around two band wheels whose radii are 5 ft. and 1 ft. respectively and whose centers are 8 ft. apart is a taut belt, not crossed. Find the length of the belt.

158. Outline a method for finding the length of a belt if the radii of the wheels are R and r, and the distance between their centers is d. (Belt is not crossed.)

159. Around two band wheels whose radii are 4 ft. and 1 ft. respectively and whose centers are 10 ft. apart is a taut belt, crossed. Find the length of the belt.

160. Outline a method for finding the length of a belt if the radii of the wheels are R and r, and the distance between their centers is d. (Belt is crossed.)

161. Simplify: (*a*) $\sin\left(\frac{\pi}{4} - x\right) + \cos\left(\frac{5\pi}{4} + x\right)$.

(*b*) $\tan\frac{1}{2}x\sin^2 x + \frac{1}{2}\sin 2x$.

(*c*) $\cos^{-1}0 + 2\tan^{-1}(-1)$.

162. Prove: $\sin^2 \frac{1}{2}x = \frac{1}{2} - \frac{1}{2}\cos x$;
and $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$.

163. Solve: $\sin x + \cos x = 1$.

164. Solve: $\sin\left(\frac{2\pi}{3} + x\right) + \sin\left(\frac{1}{3}\pi + x\right) = \frac{3}{2}$.

165. What is the circular measure of 78°? If the circular measure of a central angle is $1\frac{1}{2}$, what is the length of the arc it will intercept on a circle whose radius is 6 ft.?

166. If x is the circular measure of an angle of 30°, what is the value of $x \tan x$? What is the value of $\frac{x}{\sin x}$? Of $\frac{\cos x}{x}$?

167. Express the tangent of an angle in terms of each of the other functions.

168. Prove the relations between the functions of $\frac{3}{2}\pi - x$ and x.

169. Prove: $\sin\left(\frac{1}{3}\pi + x\right) - \sin\left(\frac{1}{3}\pi - x\right) = \sin x$.

170. Prove: $\frac{1 + \cos x + \cos 2x}{\cos x} = \frac{\sin x + \sin 2x}{\sin x}$.

171. Solve: $3\tan^2\theta + 8\cos^2\theta = 7$ for all values of θ less than 360°.

172. Show that $\sin^2 30° : \sin^2 45° : \sin^2 60° : \sin^2 90°$ as $1:2:3:4$.

SUMMARY OF FORMULAS

§ 12. $\sin A = \dfrac{1}{\csc A}$; $\csc A = \dfrac{1}{\sin A}$; $\sin A \cdot \csc A = 1.$

$\cos A = \dfrac{1}{\sec A}$; $\sec A = \dfrac{1}{\cos A}$; $\cos A \cdot \sec A = 1.$

$\tan A = \dfrac{1}{\cot A}$; $\cot A = \dfrac{1}{\tan A}$; $\tan A \cdot \cot A = 1$

§ 13. $\sin (90° - A) = \cos A$; $\cos (90° - A) = \sin A.$

$\tan (90° - A) = \cot A$; $\cot (90° - A) = \tan A.$

$\sec (90° - A) = \csc A$; $\csc (90° - A) = \sec A.$

§ 15. $\sin^2 A + \cos^2 A = 1$; $\sin A = \sqrt{1 - \cos^2 A}$;

$\cos A = \sqrt{1 - \sin^2 A}.$

$\tan^2 A + 1 = \sec^2 A$; $\tan A = \sqrt{\sec^2 A - 1}$;

$\sec A = \sqrt{\tan^2 A + 1}.$

$\cot^2 A + 1 = \csc^2 A$; $\cot A = \sqrt{\csc^2 A - 1}$;

$\csc A = \sqrt{\cot^2 A + 1}.$

$\tan A = \dfrac{\sin A}{\cos A}.$ $\cot A = \dfrac{\cos A}{\sin A}.$

§ 20.

	0°	30°	45°	60°	90°
sine	0	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$	1
cosine	1	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	0
tangent	0	$\frac{1}{3}\sqrt{3}$	1	$\sqrt{3}$	∞
cotangent	∞	$\sqrt{3}$	1	$\frac{1}{3}\sqrt{3}$	0
secant	1	$\frac{2}{3}\sqrt{3}$	$\sqrt{2}$	2	∞
cosecant	∞	2	$\sqrt{2}$	$\frac{2}{3}\sqrt{3}$	1

§§ 45 and 46. Any function of $(90° \pm x)$ or of $(270° \pm x)$ $= \pm$ the co-function of x.

Any function of $(180° \pm x)$ or of $(360° \pm x)$ $= \pm$ the same function of x.

§ 53. $\angle x = \frac{a}{r}$ radians. [$x =$ the angle expressed in radians; $a =$ the length of the arc; $r =$ the length of the radius.]

§ 54. To change radians to degrees, multiply by $\frac{180}{\pi}$.

To change degrees to radians, multiply by $\frac{\pi}{180}$.

§ 56.

1. $\sin(x+y) = \sin x \cos y + \cos x \sin y.$
2. $\cos(x+y) = \cos x \cos y - \sin x \sin y.$
3. $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}.$
4. $\cot(x+y) = \dfrac{\cot x \cot y - 1}{\cot y + \cot x}.$

§ 59.

5. $\sin 2x = 2 \sin x \cos x.$
6. $\cos 2x = \cos^2 x - \sin^2 x.$
7. $\cos 2x = 1 - 2 \sin^2 x.$
8. $\cos 2x = 2 \cos^2 x - 1.$
9. $\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}.$
10. $\cot 2x = \dfrac{\cot^2 x - 1}{2 \cot x}.$

§ 61.

11. $\sin \frac{1}{2} x = \pm\sqrt{\dfrac{1 - \cos x}{2}}.$
12. $\cos \frac{1}{2} x = \pm\sqrt{\dfrac{1 + \cos x}{2}}.$

13. $\tan \frac{1}{2} x = \pm\sqrt{\frac{1 - \cos x}{1 + \cos x}}.$

14. $\cot \frac{1}{2} x = \pm\sqrt{\frac{1 + \cos x}{1 - \cos x}}.$

§ 63.

15. $\sin (x - y) = \sin x \cos y - \cos x \sin y.$

16. $\cos (x - y) = \cos x \cos y + \sin x \sin y.$

17. $\tan (x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}.$

18. $\cot (x - y) = \frac{\cot x \cot y + 1}{\cot y - \cot x}.$

§ 65.

19. $\sin A + \sin B = 2 \sin \frac{1}{2} (A + B) \cos \frac{1}{2} (A - B).$

20. $\sin A - \sin B = 2 \cos \frac{1}{2} (A + B) \sin \frac{1}{2} (A - B).$

21. $\cos A + \cos B = 2 \cos \frac{1}{2} (A + B) \cos \frac{1}{2} (A - B).$

22. $\cos A - \cos B = - 2 \sin \frac{1}{2} (A + B) \sin \frac{1}{2} (A - B).$

§ 74. LAW OF SINES. $\frac{a}{b} = \frac{\sin A}{\sin B}; \quad \frac{b}{c} = \frac{\sin B}{\sin C};$

$\frac{a}{c} = \frac{\sin A}{\sin C}.$

§ 75. LAW OF COSINES. $a^2 = b^2 + c^2 - 2\,bc \cos A;$

$b^2 = a^2 + c^2 - 2\,ac \cos B;$

$c^2 = a^2 + b^2 - 2\,ab \cos C.$

§ 77. LAW OF TANGENTS. $\frac{a - b}{a + b} = \frac{\tan \frac{1}{2} (A - B)}{\tan \frac{1}{2} (A + B)};$

$\frac{b - c}{b + c} = \frac{\tan \frac{1}{2} (B - C)}{\tan \frac{1}{2} (B + C)};$

$\frac{a - c}{a + c} = \frac{\tan \frac{1}{2} (A - C)}{\tan \frac{1}{2} (A + C)}.$

§ 78.
$$\sin\tfrac{1}{2}A=\sqrt{\frac{(s-b)(s-c)}{bc}};$$

$$\sin\tfrac{1}{2}B=\sqrt{\frac{(s-a)(s-c)}{ac}};$$

$$\sin\tfrac{1}{2}C=\sqrt{\frac{(s-a)(s-b)}{ab}};$$

$$\cos\tfrac{1}{2}A=\sqrt{\frac{s(s-a)}{bc}};$$

$$\cos\tfrac{1}{2}B=\sqrt{\frac{s(s-b)}{ac}};$$

$$\cos\tfrac{1}{2}C=\sqrt{\frac{s(s-c)}{ab}}.$$

§ 79.
$$\tan\tfrac{1}{2}A=\frac{1}{s-a}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}};$$

$$\tan\tfrac{1}{2}B=\frac{1}{s-b}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}};$$

$$\tan\tfrac{1}{2}C=\frac{1}{s-c}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

§ 86. AREA OF A TRIANGLE.

I. Area $=\tfrac{1}{2}ab\sin C=\tfrac{1}{2}bc\sin A=\tfrac{1}{2}ac\sin B$.

II. Area $=\dfrac{a^2\sin B\sin C}{2\sin A}$, etc.

III. Area $=\sqrt{s(s-a)(s-b)(s-c)}$.

LOGARITHMS.

§ 87. $B^l = N$, the definition of a logarithm.

§ 88. $\log M \cdot N = \log M + \log N$.

§ 89. $\log \frac{M}{N} = \log M - \log N$.

§ 90. $\log M^P = P \log M$; $\log \sqrt[r]{M} = \frac{1}{r} \log M$.

§ 98. $\log_B N = \frac{\log_{10} N}{\log_{10} B}$; $\log_B N = \frac{\log_a N}{\log_a B}$.

§ 99. $\log_a B \times \log_B a = 1$.

ANSWERS

Exercise 5, p. 18

1. $\frac{7}{5}$. 2. 2. 3. $\frac{1}{5}$. 4. $\frac{1}{5}\sqrt{5}$. 5. $\frac{1}{x}$. 6. $\frac{2}{3}\sqrt{3}$. 7. $\sqrt{2}$. 8. $\frac{1}{2}$.

Exercise 6, pp. 19, 20

1. cos 47°.
2. sin 15°.
3. csc 77°.
4. cot 55°.
5. tan 64°.
6. sec 10°.
7. cos 60°.
8. sin 45°.
9. cos A.
10. tan x.
11. sin $(90° - \theta)$
12. cot $(90° - N)$
13. cos 39° 30′.
14. csc 21° 15′.
15. sin 79° 50′.
16. cot 17° 24′.
17. sec 47° 18′.
18. tan 7° 2′.
19. sin θ.
20. cot β.
21. csc 1°.
22. cos 89°.
23. tan 30°.
24. $\sin(10 - x)$.

Exercise 9, pp. 28, 29

1. 1.
2. $-\frac{1}{2}$.
3. $\frac{3}{2}$.
4. $\frac{1}{2}\sqrt{3} + 3$.
5. $-\frac{1}{3}\sqrt{3}$.
6. 2.
7. $4\sqrt{2} - 1$.
8. $\frac{1}{2} - \frac{3}{2}\sqrt{3}$.
9. $\frac{3 + \sqrt{3}}{6}$.
10. $2 - \sqrt{2}$.
11. $2\sqrt{2} - \sqrt{3}$.
12. 1.
13. $\frac{3 + 4\sqrt{3}}{2}$.
14. $1 + \sqrt{3} - \sqrt{2}$.
15. $\sqrt{2} - 1 + \frac{3}{2}\sqrt{3}$.
16. $\frac{2 + \sqrt{3}}{4}$.
17. $\frac{\sqrt{6} + \sqrt{2}}{4}$.
18. $\frac{\sqrt{6} - \sqrt{2}}{4}$.
19. $-2 - \sqrt{3}$.
20. $\frac{1}{3}\sqrt{3}$.
21. $\frac{1}{2}$.
22. $-\frac{1}{2}$.
23. $\frac{1}{2}\sqrt{2}$.
24. $\frac{1}{2}$.
25. $-\sqrt{3}$.
26. $-\frac{3}{4}$.
27. $\frac{1}{4}$.
28. $3\frac{1}{4}$.
29. $\frac{3}{16}$.
30. $\frac{9 + 2\sqrt{3}}{3}$.
31. $\frac{2\sqrt{6} - 3\sqrt{2}}{6}$.

Exercise 10, p. 31

1. 30°.
2. 45°.
3. 60°.
4. 0°.
5. 60°.
6. 60°.
7. 90°.
8. 45°.
9. 30°.
10. 30°.
11. 90°.
12. 90°.
13. 0°, 90°.
14. 60°, 0°.
15. 0°, 30°.
16. 60°, 45°.
17. 45°, 60°.
18. 30°, 90°.
19. 60°, 0°.
20. 60°, 0°.

21. 30°, 90°.
22. 0°, 45°.
23. 90°, 45°.
24. 30°.
25. 18°.
26. 18°.
27. 11° 15′.
28. 10°.
29. 36°.
30. 60°.
31. 20°.
32. 33° 20′.
33. 20°.

Exercise 11, pp. 32, 33

1. $a = 18$ ft., $c = 30$ ft.
2. $a = 14$ in., $b = 7\sqrt{21} = 32.07$ in.
3. $b = 29.4$ yd., $c = 31.98$ yd.
4. $c = 10$ m., $a = 5\sqrt{3}$ m.
5. $b = 48$ m., $c = 24\sqrt{5}$ m.
6. $a = 7.5$, $b = 10$.
7. $b = 100$, $c = 260$.
8. $a = 300$, $c = 780$.
9. $a = 9.68^{+}$, $b = 2.5$.
10. $a = 8\sqrt{2}$, $c = 12$.

Exercise 12, pp. 33, 34

1. $a = 50\sqrt{3}$ ft., $b = 50$ ft.
2. $a = 8$ in., $c = 8\sqrt{2}$ in.
3. $a = 7.5$ in., $b = 7.5\sqrt{3}$ in.
4. $a = 225$ m., $b = 225\sqrt{3}$ m.
5. $b = .6\sqrt{3}$ yd., $c = 1.2\sqrt{3}$ yd.
6. $b = 10\sqrt{3}$ m., $c = 20\sqrt{3}$ m.
7. Base $= 48\sqrt{3}$, leg $= 48$.
8. $10\sqrt{3}$.
9. $12\sqrt{3}$.
10. $3\sqrt{2}$; 3.
11. $c = 20$, $b = 10$, $p = 5$, $p' = 15$, $h = 5\sqrt{3}$.
12. $a = 200\sqrt{3}$, $b = 200$, $c = 400$, $p = 100$, $h = 100\sqrt{3}$.
13. $a = 48$, $b = 16\sqrt{3}$, $c = 32\sqrt{3}$, $p = 8\sqrt{3}$, $p' = 24\sqrt{3}$.
14. $OB = \dfrac{AB}{\sin x}$, or $= \dfrac{OA}{\cos x}$, or $= \dfrac{BC}{\tan z}$; or $= OC \cdot \cos z$, etc.
15. $OB = OC \cdot \cos z$, or $= \dfrac{BC}{\tan z}$, etc.

21. 30°.
22. 45°.
23. 30°.
24. 60°.
25. 45°.
26. 60°.

Exercise 13, pp. 37-41

1. 64.28 ; 76.6.
2. 7.27+ ; 21.28.
3. 54.95 ; 58.47+.
4. 5.718 ; 8.806.
5. 170.144 ft.
6. 357.54 yd.
7. 916.6 ft.
8. 21.13 ft. ; 22.658 ft.
9. 38.83 ; 36.93.
10. 78.32 ; 50.846.
11. 13.45 ; 18+.
12. 60.38 ; 102.42.
13. 930.86 ft.
14. 548.77 ft.
15. 257.23 ft.
16. 133.95 ft.
17. 2.309 ; 4.618.
18. 34.63.
19. 10.207 ; 8.258.
20. 9.88 ; 11.624.
21. 6.928 ; 6.928.
22. 25.89 ; 24.62.
23. 4.205 ; 2.599.
24. 19.32 ; 18.66.
25. 4.83 ; 2.588.
26. 3.105 ; 1.607.
27. 14.267 ; 9.27.
28. 63.296 ft.
29. 527.53 yd.
30. 1847.38 yd.
31. 194.85 ft.
32. 269.2 ft.
33. 41.45 ft.
34. 313.15 ft.
35. 308.38 ft.
36. $h = \dfrac{a \cdot \tan \delta}{\tan \gamma - \tan \delta}$; $d = \dfrac{a}{\tan \gamma - \tan \delta}$;
37. 567.788 ft.
38. $h = \dfrac{d \cdot \tan \theta \cdot \tan \phi}{\tan \theta + \tan \phi}$.
39. $h = \dfrac{m \cdot \tan \alpha \cdot \tan \beta}{\tan \beta - \tan \alpha}$.

Exercise 14, pp. 43, 44

1. $b = 41.182$; $c = 88.204$.
2. $b = 23.559$; $c = 25.593$.
3. $a = 250.02$; $b = 575$.
4. $A = 34^\circ 22' 9''$; $b = 51.176$.
5. $A = 33^\circ 8' 56''$; $c = 49.925$.
6. $a = 1.7653$; $c = 1.9199$.
7. $a = 10.611$; $b = 15.459$.
8. $A = 45^\circ 49' 30''$; $b = 17.003$.
9. $b = 206.07$; $c = 456.63$.
10. $A = 53^\circ 4' 18''$; $a = .11655$.
11. $A = 33^\circ 43' 11''$; $c = 5863.4$.
12. $a = .57433$; $c = .91896$.
13. $a = 1.1185$; $b = 1.3018$.
14. $A = 38^\circ 47' 2''$; $c = 15.029$.
15. $A = 66^\circ 40' 39''$; $a = 8.9534$.
16. $b = 4.89767$; $c = 5.53014$.
17. $a = 380.775$; $c = 392.018$.
18. $A = 62^\circ 13' 12''$; $c = 82.414$.
19. $a = 282.06$; $b = 154.82$.
20. $a = .56753$; $b = .63707$.
21. $A = 23^\circ 59' 58''$; $b = .0654567$.
22. $A = 44^\circ 33'$; $a = 7.991$.
23. $b = 1323.8$; $c = 2051.95$.
24. $A = 53^\circ 46' 42''$; $c = 8.8016$.
25. 5.84988 in.
26. 89.164 ft.
27. 213.58 ft.
28. 88.05 ft.
29. 50° 28′ 39″.
30. 409.95 ft.
31. Leg $= 14.6247$; $\angle = 35^\circ 53' 56''$.
32. Leg $= .48865$; $\angle = 39^\circ 49' 22''$.

Exercise 15, p. 45

1. $c = 372.346$; $h = 202.936$.
2. $c = 1.1317$; $h = .66933$.
3. $a = 1.942$; $h = 1.79387$.
4. $A = 53^\circ 2' 18''$; $h = 15.5528$.
5. $a = 7.27083$; $h = 6.847$.
6. $c = .0076325$; $h = .0076412$.
7. $a = .11852$; $c = .15075$.
8. $A = 57^\circ 18' 23''$; $c = 1006.28$.
9. 11° 28′ 42″.
10. 4.279 in.
11. $r = 60.676$; $s = 88.168$.
12. $R = 92.704$; $s = 108.98$.
13. $R = 97.992$; $r = 90.534$.
14. $s = 20.0638$; $r = 37.438$.
15. $s = 2.70888$; $r = 7.6814$.
16. $R = 5.06233$; $s = 1.58383$.
17. $R = 128.144$; $r = 125.68$.
18. $\frac{1}{2} C = 25^\circ 42' 51''$; $R = 110.992$; $s = 96.314$.

Exercise 16, p. 46

1. 6.045.
2. .33822.
3. 1.8447.
4. 30253.
5. 86.88
6. .00010253.
7. 34.6746.
8. 202.968.
9. 14.533.
10. .27576.
11. 951.065.
12. 1325.47.
13. 6924.67.
14. 16874.8.
15. 83174.
16. 16518.1.

Exercise 21, pp. 57, 59

17. 1.
18. 0.
19. -2.
20. 3.
21. 2.
22. $2\frac{1}{3}$.
23. 2.
24. $-\frac{1}{2}$.
25. 0.
26. $-a-b$.
27. No.
28. No.
29. Yes.
30. No.
31. Yes.
32. No.
33. No.
34. No.
35. No.
36. No.
37. Yes.
38. Yes.
39. 1 and -1; 1 and -1.
40. $\begin{cases} \sec > 1 \text{ and } < -1. \\ \csc > 1 \text{ and } < -1. \end{cases}$
41. May have *any* real value

Exercise 25, pp. 67–69

1. 30°.
2. 310°.
3. 260°.
4. 310°.
5. 160°.
6. 340°.
7. 240°.
8. 120°.
9. 60°.
10. 280°.
11. 45°.
12. 30°.
13. sin 7°.
14. $-$ cot 10°.
15. $-$ cos 15°.
16. $-$ csc 18°.
17. tan 42°.
18. $-$ sec 26°.
19. $-$ cot 25°.
20. sin 40°.
21. $-$ sin 10°.
22. $-$ cot 15°.
23. cos 24°.
24. $-$ cos 33°.
25. $-$ tan 9°.
26. $-$ sin 20°.
27. $-$ sin 40°.
28. $-$ tan 15°.
29. csc 41°.
30. $-$ sec 44°.
31. cot 15°.
32. cos 35° 10′.
33. $-$ cos 23° 52′.
34. cot 19° 30′.
35. cot 25° 20′.
36. csc 23° 40′.
37. $-$ csc 39° 10′.
38. $-$ sin 40° 50′.
39. $-$ cos 39° 40′.
40. $-\sin A$.
41. $-\sin\theta$.
42. $\cot x$.
43. $-\cos\theta$.
44. $-\cos\phi$.
45. $\tan\theta$.
46. $\cos\theta$.
47. $\tan x$.
48. $-\sin x$.
49. $\frac{1}{2}$.
50. $-\frac{1}{2}$.
51. $-\sqrt{3}$.
52. $\frac{1}{2}\sqrt{2}$.
53. $-\frac{1}{2}\sqrt{2}$.
54. -1.
55. $-\frac{1}{2}\sqrt{2}$.
56. $-\frac{1}{2}\sqrt{3}$.
57. $\frac{1}{2}\sqrt{3}$.
58. -2.
59. $\sqrt{3}$.
60. $-\frac{1}{2}\sqrt{3}$.
61. $-\sqrt{2}$.
62. $\frac{1}{2}$.
63. $-\sqrt{3}$.
64. $-\frac{1}{2}$.
65. $\sqrt{3}$.
66. $-\sqrt{2}$.
67. $-\frac{1}{2}\sqrt{2}$.
68. $\frac{1}{2}\sqrt{3}$.
69. $-\sqrt{3}$.
70. $\frac{1}{2}\sqrt{3}$.
71. -2.
72. $-\frac{1}{3}\sqrt{3}$.
73. 0.
74. 0.
75. 0.
76. $2\cot A$.
77. 1.
78. -1.
79. 1.
80. 1.
81. -1.
82. 0.
83. 2.
84. $-\sqrt{3}$.
85. $-\frac{3}{4}$.
86. $\frac{1}{2}$.
87. $\frac{1}{6}+\frac{4}{3}\sqrt{3}$.
88. $\frac{1}{4}\sqrt{2}-\frac{1}{3}\sqrt{3}$.
89. .58+; .72; $-$.58+; $-$.81.
90. $-$.94; 2.75; $-$.36.
91. 1.55; $-$.64; $-$1.19.
92. .61; .79; $-$.79; $-$1.28.

Exercise 26, pp. 70, 71

1. 45°, 135°, 225°, 315°.
2. 90°, 270°.
3. 45°, 135°, 225°, 315°.
4. 30°, 150°, 210°, 330°.
5. 60°, 120°, 240°, 300°.
6. 30°, 150°, 210°, 330°.
7. 60°, 120°, 240°, 300°.
8. 60°, 120°, 240°, 300°.
9. 0°, 180°.
10. 60°, 120°, 240°, 300°.

11. 30°, 150°, 210°, 330°.
12. 0°, 90°, 180°.
13. 30°, 150°, 210°, 330°.
14. 60°, 120°, 240°, 300°.
15. 0°, 45°, 180°, 315°.
16. 30°, 150°.
17. 0°, 120°, 180°, 240°.
18. 15°, 75°, 105°, 165°, 195°, 255°, 285°, 345°.
19. 20°, 40°, 80°, 100°, 140°, 160°, 200°, 220°, 260°, 280°, 320°, 340°.
20. 10°, 50°, 70°, 110°, 130°, 170°, etc.
21. 18°, 54°, 90°, 126°, 162°, 198°, etc.
22. $11\frac{1}{4}^\circ$, $33\frac{3}{4}^\circ$, $56\frac{1}{4}^\circ$, $78\frac{3}{4}^\circ$, $101\frac{1}{4}^\circ$, etc.
23. 120°, 240°. **24.** 90°, 270°. **25.** 135°. **26.** 45°, 225°, 315°.
27. 0°, 30°, 90°, 150°, 210°, 270°, 330°.
28. 90°, 135°, 225°, 270°.
29. 120, 300°.
30. 60°*, 120°*, 240°, 300°.
31. 30°*, 60°*, 120°, 150°, 210°*, 240°*, 300°, 330°. **32.** 210°, 330°.
33. 30°, 60°, 120°, 150°, 210°, 240°, 300°, 330°.
34. 60°, 90°, 120°, 240°, 270°, 300°.
35. 0°, 120°, 240°.
36. 30°, 150°, 270°.
37. 90°, 210°, 270°, 330°.
38. 0°, 45°, 135°, 180°.
39. 45°, 135°, 225°, 315°.
40. 30°, 330°.
41. 30°, 150°.
42. 60°, 180°, 300°.
43. 45°, 135°, 225°, 315°.
44. 45°, 225°.
45. 60°, 300°.
46. 30°, 45°, 150°, 225°.
47. 150°.
48. 135°, 225°.
49. 120°, 300°.
50. 60°, 240°.
51. 150°, 210°.
52. 140°.
53. 63°.
54. 243°.
55. 297°.
56. 320°.
57. 145°.
58. 260°.
59. 215°, 325°.
60. 35°, 325°.
61. 35°, 215°.
62. 50°, 130°.
63. 50°, 310°.
64. 105°, 285°.
65. 58°, 122°.
66. 122°, 238°.
67. 230°, 310°.
68. 35°, 145°.
69. 38°, 308°.
70. 10°, 350°.
71. 165°, 345°.
72. 80°, 260°.
73. 115°, 245°.
74. 63°, 297°.

Exercise 27, pp. 75, 76

8. $3\frac{1}{3}$ r.; $\frac{1}{2}$ r.; $\frac{1}{6}$ r.; .025 r.
9. 12 ft.; 6 ft.; 3 ft.; 24 ft.; 18 ft.; $3\frac{8}{11}$ ft.
10. 12 in.; 9 in.; 4 in.; 18 in.; 18.8 in.; 6.4 in.; 14 in.; $2\frac{1}{4}$ in.
11. 9 yd.
12. $\frac{\pi}{8}$; $\frac{3\pi}{8}$; $\frac{7\pi}{16}$; $\frac{9\pi}{16}$; $\frac{27\pi}{40}$.
13. $3\frac{6}{7}$ ft.; $\frac{1}{2}$ mi.; $4\frac{1}{2}$ m.; 1.53 yd.

Exercise 30, pp. 82, 83

1. $\frac{1}{4}(\sqrt{6}+\sqrt{2})$.
2. $2+\sqrt{3}$.
3. $2-\sqrt{3}$.
4. $\frac{1}{4}(\sqrt{6}+\sqrt{2})$.
5. $\frac{1}{4}(\sqrt{2}-\sqrt{6})$.
6. $-2-\sqrt{3}$.
15. $\frac{4}{5}$.
16. $\frac{1}{10}(4-3\sqrt{3})$.

---

* Will not satisfy the equation using the given sign of the radical.

17. $\frac{7}{25}$.
18. $\frac{3}{4}$.
19. $\frac{63}{65}$.
20. $\frac{16}{65}$.
21. $\frac{63}{16}$.
22. $\frac{16}{63}$.
23. $-\frac{1}{10}\sqrt{2}$.
24. $\frac{7}{10}\sqrt{2}$.
25. $-\frac{1}{7}$.

Exercise 31, pp. 84, 85

11. $\frac{24}{25}$.
12. $\frac{7}{25}$.
13. $\frac{24}{7}$.
14. $\frac{7}{24}$.
15. $-\frac{24}{25}$.
16. $-\frac{7}{25}$.
17. $\frac{24}{7}$.
18. $\frac{7}{24}$.
19. $\frac{4}{5}$.
20. $\frac{3}{5}$.
21. $\frac{4}{3}$.
22. $\frac{3}{4}$.

Exercise 32, pp. 87, 88

3. $\frac{1}{4}(\sqrt{6}-\sqrt{2})$.
4. $\frac{1}{4}(\sqrt{6}+\sqrt{2})$.
5. $2-\sqrt{3}$.
6. 1.
11. $\frac{1}{10}\sqrt{2}$.
12. $\frac{7}{10}\sqrt{2}$.
13. $\frac{1}{7}$.
14. 7.
15. $\frac{5}{\sqrt{26}}$.
16. $\frac{1}{\sqrt{26}}$.
17. 5.
18. $\sqrt{26}$.
19. $\frac{\sqrt{2}+1}{\sqrt{6}}$.
20. $\frac{1-\sqrt{2}}{\sqrt{6}}$.
21. $-3-2\sqrt{2}$.
22. $2\sqrt{3}-\sqrt{6}$.
23. $A > 360°$.
24. $A > 180°$, etc.
25. A in II, etc.
26. $A > 180°$, etc.

32. $B = 89° 57' 14''$.
33. $A = 89° 59' 42''$.
34. $B = 89° 59' 50''$.

Exercise 33, pp. 89, 90

8. $\frac{11}{15}$.
9. $\frac{14}{15}$.
10. $\frac{11}{14}$.
11. $\frac{14}{15}$.
12. $\frac{11}{15}\sqrt{2}$.
13. $\frac{11}{15}\sqrt{2}$.
14. $\frac{17}{31}$.
15. $\frac{25\sqrt{2}}{81}$.
16. $\frac{2\sqrt{2}-2}{3\sqrt{5}}$.
17. $\frac{-4\sqrt{2}-1}{3\sqrt{5}}$.
18. $\frac{10\sqrt{2}-18}{31}$.
19. $\frac{1}{2}(\sqrt{10}+\sqrt{5})$.
20. $\frac{5}{13}$.
21. $\frac{1}{26}(12\sqrt{3}-5)$.
22. $\frac{5\sqrt{3}+12}{5-12\sqrt{3}}$.
23. $-\frac{120}{169}$.
24. $\frac{3}{13}\sqrt{13}$.
25. $\frac{3}{169}\sqrt{13}$.

Exercise 34, pp. 92, 93

1. $2\cos 8x\cdot\sin x$.
2. $2\sin 3A\cdot\cos 2A$.
3. $2\cos 2\theta\cdot\cos\theta$.
4. $2\cos 6m\cdot\sin m$.
5. $2\sin 4A\cdot\sin 2A$.
6. $-2\sin 3\theta\cdot\sin 2\theta$.
7. $2\cos 7x\cdot\cos 2x$.
8. $\sqrt{3}\cdot\sin 50°$.
9. $2\sin x\cdot\cos\frac{1}{2}x$.
10. $2\cos\theta\cdot\cos$ [illegible] θ.
11. $2\sin 16°\cdot\sin 1°$.
12. $2\cos(2x+y)\cdot\sin(2x-y)$.
28. $2\cos 20°\cdot\sin A$.
29. $2\sin 2A\cos(2B-A)$.
30. $\sqrt{2}\cdot\cos(18°+x)$.
31. $2\sin 50°\cdot\sin\theta$.
32. $\cot\frac{1}{2}(A+B)\cdot\tan\frac{1}{2}(A-B)$.
33. $\tan\frac{1}{2}(x+y)$.
34. $-\cot\frac{1}{2}(\theta+\phi)$.

Exercise 36, pp. 94–98

47. $2 \csc 2x$.

48. $\dfrac{\sin(A+B)}{\cos A \cos B}$.

49. $2 \sin 23^\circ \cos 9^\circ$.

50. $\dfrac{\cos(\theta + \theta')}{\sin\theta \cos\theta'}$.

51. $\tan^2 x$.

52. $\dfrac{-\cos(x+y)}{\cos x \cos y}$.

53. $\dfrac{\cos 5^\circ}{\cos 10^\circ \cos 15^\circ}$.

54. $\dfrac{\sin(B-A)}{\sin A \cos B}$.

55. $\tan x \tan y$.

56. $\tan 23^\circ$.

57. $\frac{1}{10}\sqrt{10}$.

58. $\frac{1}{6}(1 + 2\sqrt{6})$; $\frac{4}{7}\sqrt{2}$; $\frac{1}{3}\sqrt{3}$; $-\frac{1}{3}\sqrt{6}$.

59. $\frac{119}{169}$; -5.

60. $\frac{2}{9}(1 + \sqrt{10})$; $\dfrac{2\sqrt{10}+2}{4\sqrt{2}-\sqrt{5}} = \frac{2}{3}(\sqrt{5}+\sqrt{2})$.

61. $\frac{1}{25}(4\sqrt{21} + 6$; $\frac{1}{50}(2\sqrt{10} - 3\sqrt{210})$.

62. $-\frac{1}{3}$; $\frac{4}{7}\sqrt{2}$.

63. $\frac{3}{4}$.

64. $\frac{1}{4}\sqrt{7}$; $\frac{1}{4}\sqrt{14}$.

65. $1 - 2x^2$.

66. $\dfrac{m}{\sqrt{1+m^2}}$.

67. $-2a\sqrt{1-a^2}$; $1 - 2a^2$.

68. $\frac{7}{25}$; $\frac{24}{7}$; $\frac{3}{4}$; $\frac{336}{625}$.

69. $\frac{3}{4}$; $\frac{4}{5}$; $\frac{1}{3}$; $\frac{7}{25}$.

70. $-\frac{4}{3}$; $-\frac{3}{5}$; 2; $-\frac{7}{25}$.

71. $-\frac{1}{5}$; $-2\sqrt{6}$; $\frac{4}{25}\sqrt{6}$.

Exercise 37, pp. 99–100

1. 0°, 60°, 180°, 300°.
2. 30°, 90°, 150°, 270°.
3. 0°, 60°, 120°, 180°, 240°, 300°.
4. 90°, 270°.
5. 60°, 180°, 300°.
6. 0°, 90°, 270°.
7. 90°.
8. 30°, 150°.
9. 30°, 150°, 270°.
10. 45°, 135°, 225°, 315°.
11. 30°, 150°, and ∠ whose sine $= -\frac{1}{3}$.
12. 180°, and ∠ whose cosine $= \frac{5}{8}$.
13. 30°, 150°, 210°, 330°.
14. 30°, 120, 150°, 240°.
15. 60°, 270°, 300°.
16. 0°, 210°, 330°.
17. 45°, 135°, 225°, 315°.
18. 0°, 180°, etc.
19. 30°, 150°, and ∠ whose sine is $\frac{1}{4}$.
20. 0°, $7\frac{1}{2}$°, $37\frac{1}{2}$°, $97\frac{1}{2}$°, $127\frac{1}{2}$°, 180°, $187\frac{1}{2}$°, etc.
21. 45°, 60°, 135°, 225°, 300°, 315°.
22. 90°, 120°, 240°, 270°.
23. 0°, 60°, 180°, 300°.
24. 30°, 150°, 210°, 330°.
25. 0°, 90°, 180°, 270°.
26. 30°, 150°, 180°, 270°.
27. 30°, 90°, 120°, 150°, 240°, 270°.
28. $67\frac{1}{2}$°, 90°, $157\frac{1}{2}$°, $247\frac{1}{2}$°, 270°, $337\frac{1}{2}$°.
29. 0°, 10°, 45°, 50°, 90°, 130°, 135°, 170°, 180°, 225°, 250°, 270°, 290°, 315°.
30. 45°, 135°, 210°, 225°, 315°, 330°.
31. $22\frac{1}{2}$°, 45°, $67\frac{1}{2}$°, 90°, $112\frac{1}{2}$°, 135°, $157\frac{1}{2}$°, 225°, etc.
32. 0°, 72°, 90°, 144°, 180°, 216°, 270°, 288°.
33. 0°, 30°, 150°, 180°, 210°, 330°.
34. 60°, 90°, 120°, 240°, 270°, 300°.
35. 30°, 60°, 120°, 150°, 210°, 240°, 300°, 330°.

Exercise 38, p. 101

1. $y = 45°, 225°$,
$x = \sqrt{2}, -\sqrt{2}$.

2. $y = 150°, 330°$,
$x = -4, 4$.

3. $\theta = 60°, 300°$,
$y = 4$.

4. $x = \dfrac{b \cos A - a \sin A}{\cos 2A}$,
$y = \dfrac{a \cos A - b \sin A}{\cos 2A}$.

5. The equations reduce to:
$x \sin \theta = 2a$; $x\sqrt{3} \cos \theta + x \sin \theta = 2b$.
$\therefore \tan \theta = \dfrac{a\sqrt{3}}{b-a}$, from which θ may be found. $\therefore x = \dfrac{2a}{\sin \theta}$.

6. Expand the two equations, add and subtract. Divide and obtain:
$\tan \theta = \dfrac{a+b}{a-b} \tan m$, whence θ is known.
$\therefore y =$ etc.

7. $T = \pm \frac{2}{3}$; $\theta = 30°$, etc.

8. $P = \pm 4\sqrt{5}$; $\cos \theta = \pm \frac{2}{5}\sqrt{5}$.

9. $P = 10$; $\tan x = \frac{1}{2}$.

Exercise 40, pp. 103, 104

1. $\frac{4}{5}$.

2. $3\sqrt{11}$.

3. $\dfrac{1}{\sqrt{1+x^2}}$.

4. $\dfrac{5}{\sqrt{21}}$.

5. $\frac{2}{5}\sqrt{5}$.

6. $2\sqrt{2}$.

11. $\frac{7}{8}$.

12. $\frac{1}{2}\sqrt{3}$.

13. $\frac{24}{25}$.

14. $\frac{1}{3}\sqrt{3}$.

15. $-\frac{4}{5}$.

16. 3.

32. $\frac{1}{25}(3\sqrt{21} + 8)$.

33. $\frac{1}{5}(9\sqrt{3} - 8\sqrt{2})$.

34. $\frac{1}{2}\sqrt{2}$.

Exercise 41, p. 105

1. $30°, 150°, \sin^{-1}(-\frac{2}{3})$.
2. $45°, 225°, \tan^{-1}(-5)$.
3. $180°, \cos^{-1}\frac{1}{3}$.
4. $\tan^{-1}\frac{3}{4}$.
5. $15°, 75°, 195°, 255°$.
6. $\tan^{-1} 2$; $\tan^{-1}(-3)$.
7. $60°, 120°, 240°, 300°, \sin^{-1}\frac{3}{4}$.
8. $0°, 90°$.
9. $30°, 150, \sin^{-1}\dfrac{-1 \pm \sqrt{5}}{4}$.
10. $0°, 90°, 180°, 270°$.
11. $x = 0, \pm\frac{1}{2}\sqrt{3}$.
12. $x = 0, \pm 1$.
13. $x = 0, \pm\sqrt[4]{\frac{3}{4}}$.

Exercise 42, pp. 109, 110

1. 68° 21′.
2. $A = 48° 54'$; $B = 69° 50'$.
3. $A = 28° 54'$; $C = 80° 34'$.
5. 55, 30, 18, 7.
6. 10.58, 3.40, 6.02, 1.16.
7. 1.0072, .5896, .2019, .2151.

Exercise 44, p. 114

1. $b = 2.79087$,
   $c = 1.63189$.
2. $a = 795$,
   $b = 567.688$.
3. $a = 59.784$,
   $c = 59.051$.
4. $b = 130.5525$,
   $c = 301.882$.
5. $b = 37.5458$,
   $c = 32.7007$.
6. $a = 2199.4$,
   $b = 1205.7$.
7. $a = 137.15$,
   $c = 112.14$.
8. $b = .06056$,
   $c = .092926$.
9. $a = 653.52$,
   $c = 1143.851$.
10. $a = .13954$.
    $b = .09787$.

Exercise 45, pp. 115, 116

1. $A = 97° 59' 21''$,
   $B = 44° 32' 39''$,
   $c = 14.7424$.
2. $B = 42° 24' 53''$,
   $C = 66° 7''$,
   $a = 111.126$,
3. $A = 77° 18' 24''$,
   $C = 39° 5' 36''$,
   $b = 579.95$.
4. $A = 28° 26' 39''$,
   $C = 70° 22' 51''$,
   $b = 74.0667$.
5. $A = 88° 17' 39''$,
   $B = 47° 15' 5''$,
   $c = .089438$.
6. $A = 42° 34' 27''$,
   $B = 116° 5' 45''$,
   $c = 17.046$.
7. $B = 45° 39' 56''$,
   $C = 72° 50' 28''$,
   $a = 587.94$.
8. $A = 52° 3' 19''$,
   $C = 56° 15' 3''$,
   $b = 469.24$.
9. $A = 49° 26' 27''$,
   $B = 28° 29' 51''$,
   $c = .009824$.
10. $B = 20° 27' 1''$,
    $c = 36° 15' 7''$,
    $a = 998.35$.

Exercise 46, p. 117

1. $A = 44° 38' 52''$,
   $B = 62° 40' 12''$,
   $C = 72° 41' 2''$.
2. $A = 44° 48' 12''$,
   $B = 52° 55' 56''$,
   $C = 82° 15' 50''$.
3. $A = 83° 14' 58''$,
   $B = 45° 9' 24''$,
   $C = 51° 35' 38''$.
4. $A = 49° 24'$,
   $B = 58° 38' 11''$,
   $C = 71° 57' 49''$.
5. $A = 57° 37' 4''$,
   $B = 70° 28' 4''$,
   $C = 51° 54' 46''$.
6. $A = 37° 11' 6''$,
   $B = 58° 29' 24''$,
   $C = 84° 19' 30''$.

7. $A = 21° 58' 16''$,
$B = 98° 21' 12''$.

8. $A = 21° 16' 28''$,
$B = 43° 35' 22''$.

9. $A = 43° 35' 16''$,
$B = 58° 28'$.

10. $B = 31° 47' 7''$,
$C = 90°$.

Exercise 47, p. 119

1. One $\triangle$.
2. Impossible.
3. Two $\triangle$s.
4. One rt. $\triangle$.
5. One isos. $\triangle$.
6. Impossible.
7. One $\triangle$.
8. Two $\triangle$s.
9. One $\triangle$.
10. Two $\triangle$s.
11. Impossible.
12. Impossible.
13. One rt. $\triangle$.
14. Two $\triangle$s.
15. One $\triangle$.

Exercise 48, p. 121

1. $B = 12° 13' 34''$,
$C = 146° 15' 26''$,
$c = 636.09$.

2. $A = 90°$,
$C = 32° 22' 43''$,
$c = 83.7$.

3. $A = 93° 9' 13''$,
$A' = 5° 46' 33''$,
$C = 46° 18' 40''$,
$C' = 133° 41' 20''$,
$a = 69.456$,
$a' = 6.99906$.

4. $A = 25° 32' 51''$,
$B = 39° 57' 18''$,
$b = 11.1277$.

5. $A = 44° 42' 40''$,
$C = 90°$,
$a = 275.744$.

6. Impossible.

7. $A = 132° 12'$,
$B = 10° 54' 58''$,
$a = 5680.05$.

8. $A = 70° 9'$,
$A' = 109°, 51'$,
$C = 57° 26' 36''$,
$C' = 17° 44' 36''$,
$c = .28824$,
$c' = .1042$.

9. Impossible.

10. Impossible.

11. $A = 68° 58' 24''$,
$A' = 111° 1' 36''$,
$B = 48° 43' 48''$,
$B' = 6° 40' 36''$,
$b = 294.1$,
$b' = 45.496$.

12. $B = 13° 26' 10''$,
$C = 90°$,
$b = 5.7897$.

Exercise 49, pp. 121–124

1. $b = 79.862$,
$c = 36.9675$.

2. $B = 77° 52' 21''$,
$C = 58° 99' 21''$,
$a = 30.942$.

3. $A = 41° 25' 22''$,
$B = 72° 5' 16''$.

4. $B = 29° 5' 21''$,
$C = 80° 6' 14''$,
$c = 19.272$.

5. $A = 52^\circ\ 40'\ 38''$,
$B = 55^\circ\ 50'\ 28''$.
6. $A = 96^\circ\ 39'\ 52''$,
$B = 44^\circ\ 32'\ 44''$,
$c = 270.32$.
7. $A = 99^\circ\ 16'\ 27''$,
$A' = 4^\circ\ 6'\ 35''$,
$B = 42^\circ\ 25'\ 4''$,
$B' = 137^\circ\ 34'\ 56''$,
$a = 41.142$,
$a' = 2.9876$.
8. $a = 420.71$,
$c = 660.9$.
9. $A = 25^\circ\ 32'\ 10''$,
$B = 67^\circ\ 50'\ 38''$.
10. Impossible.
11. $b = 760.4$,
$c = 318.979$.
12. $A = 37^\circ\ 24'\ 26''$,
$A' = 142^\circ\ 35'\ 34''$,
$C = 113^\circ\ 56'\ 46''$,
$C' = 8^\circ\ 44'\ 38''$,
$c = 26.5253$,
$c' = 4.41211$.
13. 54° 27′ 45″.
14. 39.503; 51.939.
15. 35° 17′ 50″.
16. 49.124 rd.; 69.89 rd.
17. 25.416; 71.133.
18. 134.4 mi.
19. 61° 31′ 10″.
20. 55.296 yd.
21. 48.812 ft.; 40.333 ft.
22. 179.367 ft.
23. 43.288 in.
24. 456.2 ft.
25. 146.74.
26. 120.078 yd.
27. 52° 49′ 52″.
28. 6.53 in.
29. 51° 4′ 18″ E. of N. or W. of N.
30. 138.12 ft.
31. 45.858; 11.068.
32. 30.216 ft.
33. 491.34 yd.; 417.57 yd.
34. 18° 0′ 54′.
36. 64.95 yd.; 37.5 yd.
37. 155.23 ft.; 40.276 ft.
38. 6.4867; 10.796.

Exercise 50, pp. 125, 126

3. 1527.85.
4. 8.751.
5. 2228.4.
6. 50.8533.
7. .51041.
8. .00023775.
9. 2.0339.
10. 192.34,
34.252.
11. 23.417.
12. 30.838.
15. 4718.9.
16. 269.35.
17. 56° 40′ 30″.
18. 12.997.
19. 76.2183.
20. 1207.44.
21. 54.026.

Exercise 53, pp. 134, 135

1. 28.7327.
2. 1.83633.
3. 73.62.
4. 3.5955.
5. −3.63483.
6. 29.244.
7. 3.8265.
8. 13.474.
9. .5109.
10. −20.288.
11. 10.6312.
12. 2.7778.
13. .18344.
14. .13187.
15. 5.0919.
16. 2.3834.
17. 19.429.
18. .017113.
19. 12.644.
20. 13.285.

Exercise 54, p. 136

1. 2.128+.
2. 5.145+.
3. 1.156+.
4. 1.025+.
5. 1.472+.
6. 1.184−.
7. 1.3505−.
8. .7805−.
9. .5265+.
10. − 2.286+.
11. − .6129−.
12. .4699−.
13. 1.8827−.
14. 1.951+.
15. 2.453+.
16. 2.623+.
17. 2.5009+.
18. 2.335+.
19. − .04105+
20. − 1.539+
21. − 8.178+
22. .234+.
23. 1.4307−.
24. 3.321+.

Exercise 55, pp. 137, 138

1. $7387.33.
2. $9990.75.
3. $8374.60.
4. $2405.
5. 15.718 yr.
6. 7.573 yr.
7. Nearly $5\frac{1}{4}$%.
8. Nearly $3\frac{1}{2}$%.
9. 23.44 yr.; 17.67 yr.; 14.206 yr.; 11.9− yr. and 10.246 yr.

Exercise 56, pp. 139, 140

Results of examples 1–15 can be verified from the tables

16. 2.
17. $\frac{7}{8}$.
18. $\frac{1}{64}$.
19. $\frac{3}{4}$.
20. $\frac{2}{3}$.
21. $-\frac{3}{4}$.
22. $\frac{6}{5}$.
23. 2.
24. $\frac{4}{3}$.
25. $-\frac{3}{2}$.
26. $\frac{2}{3}$.
27. 27.
28. 243
29. 32
30. $\frac{243}{32}$.

Exercise 58, pp. 141, 142

54. 43° 14′.
55. 27° 17′.
56. 65° 50′ 54″,
    9.65167 − 10.
57. 9.98605 − 10,
    9.39705 − 10.
58. 41° 57′.
59. 11° 20′ 58″.
60. 27° 11′ 44″.
61. 57° 37′ 30″.
62. 23° 36′ 30″.
63. 58° 48′ 34″.

Exercise 59, pp. 143-152

1. 2.23.
2. 1.4.
3. $-\frac{1}{2}\sqrt{3}$; $-\frac{1}{2}$.
6. 44° 2′ 54″.
7. $-\frac{2\sqrt{2}(7\sqrt{3}+2)}{45}$.
8. 77° 41′ 36″,
   44° 55′ 14″.
13. 0°, 90°, 180°, 270°.
14. $A = 41°\ 58'\ 55''$.
15. $\frac{1}{2}$; $-\frac{1}{3}$.
16. $\frac{1}{2}\sqrt{2+\sqrt{2}}$; $\frac{1}{2}\sqrt{2-\sqrt{2}}$
17. No.
18. 48° 43′ 23″.
20. 18.56 in.,
    27° 15′ 19″,
    62° 44′ 41″.
21. 1.817 mi.
22. 30°, 150°.
26. $\frac{\sqrt{3}(\sqrt{5}+1)}{6}$.
28. 315.87 ft.
31. 1.21786.
33. $-2\sqrt{2}-\sqrt{6-2\sqrt{2}}$

35. 60°, 300°.
36. 36° 52′ 11″.
37. 77.872 ft.,
186.98 ft.,
37° 45′ 19″,
area = 9008.4 sq. ft.
38. 3083.53 mi.
40. 20°.
41. 14° 28′ 7″.
42. 45°, 120°, 225°, 240°.
44. 1.35 radians,
77° 20′ 58″.
49. 66.8 ft.
50. 48° 11′ 21″.
61. 32.3608.
65. 30°, 150°, 210°, 300°.
69. $B = 88°\ 8'\ 55''$,
$B' = 14°\ 54'\ 41''$,
$C = 53°\ 22'\ 53''$,
$C' = 126°\ 37'\ 7''$,
$b = 7.5687$,
$b' = 1.9486$.
70. $\frac{\pi}{2}$; π.
71. 19° 47′ 24″.
72. a.
74. 60°.
75. 4.1875,
49.956,
7844.8.
76. 295.07 yd.
78. 85° 18′ 18″,
68158.
80. $b = 12.977$,
$c = 10.648$.
83. $A = 26°\ 43'\ 20''$,
$B = 92°\ 25'\ 2''$,
$C = 60°\ 51'\ 42''$.
84. $b = 8.2126$,
$a = 7.5528$.
86. $-\frac{2}{3}$.
89. 95.5.
90. .366 ; .9826.
93 2966.7 ; 70° 4′ 54″.

100. 70° 52′ 40″,
47° 0′ 38″,
62° 6′ 46″.
101. $\frac{15}{2}(\sqrt{3} - 1) = 5.49$.
104. 60°, 150°, 240°, 330°.
106. 63.4 ft.
110. 1.54.
111. 20.16 ft.
113. 205° 25′ 11″,
0.43541.
114. 45°, 60°, 120°, 135°, 225°, 240°,
300°, 315°.
115. 1.989.
116. 76° 44′ 14″.
118. 51° 36′ 16″.
119. 686.17 ; 233.821.
120. 1.5480,
2.5480,
1.9084.
121. 12.944.
122. .83002.
123. 7° 14′ 55″.
124. 25° 30′ 41″.
128. 48.4478.
130. 30°, 150°, 210°, 330°.
131. 30°, 90°, 150°, 210°, 270°, 330°.
132. $\theta = 30°, 210°$; $x = \pm 2\sqrt{3}$.
133. $\frac{1}{2}\sqrt{2 - \frac{6}{13}\sqrt{13}}$, or $\frac{1}{2}\sqrt{2 + \frac{4}{13}\sqrt{13}}$;
$\frac{1}{2}(\sqrt{13} - 3)$, or $\frac{1}{3}(\sqrt{13} + 2)$.
134. 45°, 135°, 225°, 315°.
157. $\frac{2}{3}(12\sqrt{3} + 11\pi)$ ft.
159. $\frac{10}{3}(3\sqrt{3} + 2\pi)$ ft.
161. 0; $\sin x$; 0°, or 180°.
163. 0°, or 90°.
164. 30°, or 330°.
165. $\frac{13\pi}{30}$ r.; 9 ft.
166. $\frac{\pi\sqrt{3}}{18}$; $\frac{\pi}{3}$; $\frac{3\sqrt{3}}{\pi}$
171. 30°, 45°, 135°, 150°, 210°, 225°,
315°, 330°.